MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. LI.

MATHEMATICAL

QUESTIONS AND SOLUTIONS,

FROM THE "EDUCATIONAL TIMES," VIII. 339

WITH MANY

PAPERS AND SOLUTIONS

IN ADDITION TO THOSE

PUBLISHED IN THE "EDUCATIONAL TIMES,"

AND THREE

APPENDICES.

EDITED BY

W. J. C. MILLER, B.A.,

REGISTRAR OF THE GENERAL MEDICAL COUNCIL.

VOL. LI.

LONDON:

FRANCIS HODGSON, 89, FARRINGDON STREET, E.C.

1889.

*** Of this series there have now been published fifty-one Volumes, each of which contains, in addition to the papers and solutions that have appeared in the *Educational Times*, an equal quantity of new articles, and comprises contributions, in all branches of Mathematics, from most of the leading Mathematicians in this and other countries.

New Subscribers may have any of these Volumes at Subscription-prices.

LIST OF CONTRIBUTORS.

AIYAR, Professor SWAMINATHA, M.A.; Madras.
ALLEN, Rev. A. J. C., M.A.; College, Chester.
ALLMAN, Professor GEO. J., LL.D.; Galway.
ANDERSON, ALEX., B.A.; Queen's Coll., Galway.
ANTHONY, EDWYN, M.A.; The Elms, Hereford.
ARMENANTE, Professor; Pesaro.
BALL, Sir ROBT. STAWELL, LL.D., F.R.S.; Dublin.
BARTON, W. J., M.A.; Highgate, London.
BASÛ, Prof. ABINASH CHANDRA, M.A.; Calcutta.
BASÛ, Professor KALIPADA, Dacca College.
BATTAGLINI, Prof. GIUSEPPE; Univ. di Roma.
BAYLISS, GEORGE, M.A.; Kenilworth.
BEEVOR, R. J., B.A.; Regent's Park, London.
BELTRAMI, Professor; University of Pisa.
BERG, Professor F. J. VAN DEN; Delft.
BESANT, W. H., D.Sc., F.R.S.; Cambridge.
BEYENS, Professor IGNACIO, M.A.; Cadiz.
BHATTACHARYA, Prof. MUNINDRANATH, M.A.
BHUT, Professor ATH BIJAH, M.A.; Dacca.
BICKERDIKE, C.; Kippax, Yorkshire.
BIDDLE, D.; Gough Ho., Kingston-on-Thames.
BIRCH, Rev. J. G., M.A.; Bunralty, co. Clare.
BLACKWOOD, ELIZABETH, B.Sc.; Boulogne.
BLYTHE, W. H., B.A.; Egham.
BORCHARDT, Dr. C. W.; Victoria Strasse, Berlin.
BORDAGE, Prof. EDMOND; Coll. de Nantua.
BOULT, S. H., M.A.; Liverpool.
BOURNE, C. W., M.A.; College, Inverness.
BRILL, J., B.A.; St. John's Coll., Cambridge.
BROCARD, H., Chef de Bataillon; Grenoble.
BROOKS, Professor E.; Millersville, Pennsylvania.
BROWN, A. CRUM, D.Sc.; Edinburgh.
BRUNYATE, W. E.; Trinity Coll., Cambridge.
BRYAN, G. H., M.A.; Cambridge.
BUCK, EDWARD, M.A.; Univ. Coll., Bristol.
BURNSIDE, Professor W. S., M.A.; Univ., Dublin.
BURSTALL, H. J. W.; Sch. of St. John's Coll., Cam.
CAMERON, HECTOR C., M.A.; Glasgow.
CAPEL, H. N., LL.B.; Bedford Square, London.
CARMODY, W. P., B.A.; Clonmel Gram. School.
CARR, G. S., M.A.; 3 Endsleigh Gardens, N.W.
CASEY, Prof. LL.D., F.R.S.; Cath. Univ., Dublin
CASEY, W. P., M.A.; San Francisco.
CATALAN, Professor; University of Liège.
CAVALLIN, Prof., M.A.; University of Upsala.
CAVE, A. W., B.A.; Magdalen College, Oxford.
CAYLEY, A., F.R.S.; Sadlerian Professor of Mathematics in the University of Cambridge, Member of the Institute of France, &c.
CHAKRAVARTI, Prof. BYOM., M.A.; Calcutta.
CHASE, Prof., LL.D.; Haverford College.
CHRISTIE, R. W. D.; Wavertree Park.
CLARKE, Colonel A. R., C.B., F.R.S.; Redhill.
COCHEZ, Professor; Paris.
COCKLE, Sir JAMES, M.A., F.R.S.; London.
COHEN, ARTHUR, M.A., Q.C., M.P.; Holland Pk.
COLSON, C. G., M.A.; University of St. Andrews.
CONSTABLE, Rev. W. J., M.A.; Uppingham.
COTTERILL, J. H., M.A.; R. N. Coll., Greenwich.
CRAWFORD, G. E., B.A.; Clifton, Bristol.
CREMONA, Prof. LUIGI; Rome.
CROFTON, Professor M. W., F.R.S.; Dublin.
CROKE, J. O'BYRNE, M.A.; Dublin.
CULLEY, Prof., M.A.; St. David's Coll., Lampeter.
CURTIS, R., M.A., S.J.; Univ. Coll., Dublin.
DARBOUX, Professor; Paris.
DATA, Prof. PROMATHANATH, M.A.; Calcutta.
DAVIES, D. O. S., M.A.; Univ. Coll., Bangor.
DAVIS, R. F., M.A.; Endsleigh Gardens.
DAWSON, H. G., B.A.; Christ Coll., Cambridge.
DAY, Rev. H. G., M.A.; Richmond Terr., Brighton.
DE LONGCHAMPS, Professor; Paris.
DE WACHTER, Prof., M.A.; Schaarbeck.
DEY, Prof. NARENDRA LAL, M.A.; Calcutta.
DICK, G. R., M.A.; Victoria University.
DOBSON, T., B.A.; Hexham Grammar School.
D'OCAGNE, MAURICE; Cherbourg.
DROZ, Prof. ARNOLD, M.A.; Porrentruy, Berne.
DUPAIN, J. C.; Professeur au Lycée d'Angoulême.
EASTON, BELLE, B.Sc.; Lockport, New York.
EDWARD, J., M.A.; Head Mast., Aberdeen Coll.
EDWARDES, DAVID, B.A.; Dublin.
ELLIOTT, E. B., M.A.; Fell. Queen's Coll., Oxon.
ELLIS, ALEXANDER J., F.R.S.; Kensington.
ELTON, E. F., M.A.; Wellington College.
EMMERICH, Prof., Ph.D.; Mülheim-am-Ruhr.
ESSENNELL, EMMA; Coventry.
EVANS, Professor, M.A.; Lockport, New York.
EVERETT, Prof. J. D., D.C.L.; Qu. Coll., Belfast.
FICKLIN, JOSEPH; Prof. in Univ. of Missouri.
FINKEL, B. F.; Ada, Ohio.
FORTEY, H., M.A.; Clifton, Bristol.
FOSTER, F. W., B.A.; Chelsea.
FOSTER, Prof. G. CAREY, F.R.S.; Univ. Coll., Lond.
FOSTER, W. S., Hoddesdon, Herts.
FRANKLIN, CHRISTINE LADD, M.A.; Prof. of Nat. Sci. and Math., Union Springs, New York.
GALBRAITH, Rev. J., M.A.; Fell. Trin. Coll., Dublin.
GALE, KATE; Worcester Park, Surrey.
GALLATLY, W., M.A.; Highgate, London.
GALLIERS, Rev. T., M.A.; Kirstead Rect., Norwich
GALTON, FRANCIS, M.A., F.R.S.; London.
GENESE, Prof., M.A.; Univ. Coll., Aberystwith.
GERRANS, H. T., M.A.; Fel. & Tut., Wor. Coll., Oxon
GLAISHER, J. W. L., F.R.S.; Fell. Trin. Coll., Camb.
GOLDENBERG, Professor, M.A.; Moscow.
GORDON, ALICE, B.Sc.; Gloucester.
GOUDIE, W. P., Irvine House; Derby.
GRAHAM, R. A., M.A.; Trinity College, Dublin.
GREENFIELD, Rev. W. J., M.A.; Dulwich College.
GREENHILL, A. H., M.A.; Artillery Coll., W'wich.
GREENSTREET, W. J., B.A.; Cardiff College.
GREENWOOD, JAMES M.; Kirksville, Missouri.
GRIFFITHS, D. T., M.A.: Jesus College, Oxford.
GRIFFITHS, G. J., M.A.; Fell. Ch. Coll., Camb.
GRIFFITHS, J., M.A.; Fellow of Jesus Coll., Oxon.
GROSS. W., LL.D.; Bournemouth.
GROVE, W. B., B.A.; Perry Bar, Birmingham.
HADAMARD, Professor, M.A.; Paris.
HAIGH, E., B.A., B.Sc.; King's Sch., Warwick.
HALL, Professor ASAPH, M.A.; Washington.
HAMMOND, J., M.A.; Buckhurst Hill, Essex.
HANUMANTA RAU, Professor, B.A.; Madras.
HARKEMA, C.; University of St. Petersburg.
HARLEY, Rev. ROBERT, M.A., F.R.S.; Oxford.
HARRIS, H. W., B.A.; Trinity College, Dublin.
HART, H.; R. M. Academy, Woolwich.
HAUGHTON, Rev. Dr., F.R.S.; Trin. Coll., Dubl.
HENDRICKS, J. E., M.A.; Des Moines, Iowa.
HEPPEL, G., M.A.; The Grove, Hammersmith.
HERMAN, R. A., M.A.; Trin. Coll., Cambridge.
HERMITE, CH.; Membre de l'Institut, Paris.
HERVEY, F. R. J., M.A.; Worthing.
HILL, Rev. E., M.A.; St. John's College, Camb.
HIME, Col. H. W. L., R.A.,; Kilkee, India.
HINTON, C. H., M.A.; Cheltenham College.
HIRST, Dr. T. A., F.R.S.; London.
HOLT, J. R., M.A.; Trinity College, Dublin.
HOPKINS, Rev. G. H., M.A.; Stratton, Cornwall.
HOPKINSON, J., D.Sc., B.A.; Kensington.
HOROBIN, S. C.; Homerton College.
HUDSON, C. T., LL.D.; Manilla Hall, Clifton.
HUDSON, W. H. H., M.A.; Prof. in King's Coll., Lond.
JACKSON, Miss F. H.; Towson, Baltimore.
JENKINS, MORGAN, M.A.; London.
JOHNSON, Prof., M.A.; Annapolis, Maryland.
JOHNSTON, T. P., B.A.; Trin. Coll., Dublin.
JOHNSTON, W. J., M.A.; Univ. Coll., Aberystwith.
KAHN, A., B.A.; St. John's Coll., Camb.
KENNEDY, D., M.A.; Catholic Univ., Dublin.
KIRKMAN, Rev. T. P., M.A., F.R.S.; Croft Rect.
KITCHIN, Rev. J. L., M.A.; Heavitree, Exeter.
KITTUDGE, LIZZIE A.; Boston, United States.
KNISELY, Rev. U. J.; Newcomerstown, Ohio.
KNOWLES, R., B.A., L.C.P.; Tottenham.
KOEHLER, J.; Rue St. Jacques, Paris.
LACHLAN, R., B.A.; Trinity Coll., Cambridge.
LAMPE, Prof., Ed. of *Jahrb. der Math.*; Berlin.
LANGLEY, E. M., B.A.; Bedford.
LAVERTY, W. H., M.A.; late Exam. in Univ. Oxford.
LAWRENCE, E. J.; Ex-Fell. Trin. Coll., Camb.
LEGABEKE, Professor; Delft.
LEIDHOLD, R., M.A.; Finsbury Park.
LEMOINE, E.; 5, Rue Littré, Paris.
LEUDESDORF, C., M.A.; Fel. Pembroke Coll., Oxon.
LEVETT, R., M.A.; King Edw. Sch., Birmingham.
LIAGRE, General; 23 Rue Carotz, Brussels.
LONDON, Rev. H., M.A.; Pocklington.

LOWRY, W. H., M.A.; Blackrock, Dublin.
MCALISTER, DONALD, M.A., D.Sc.; Cambridge.
MCCAY, W. S., M.A.; Fell. Trin. Coll., Dublin.
MCCLELLAND, W. J., B.A.; Prin. of Santry School.
MCCOLL, HUGH, B.A.; Boulogne.
MACDONALD, W. J., M.A.; Edinburgh.
MACFARLANE, Prof. A., D.Sc.; Univ. of Texas.
MCINTOSH, ALEX., B.A.; Bedford Row, London.
MACKENZIE, J. L., B.A.; Gymnasium, Aberdeen.
MCLEOD, J., M.A.; R.M. Academy, Woolwich.
MACMAHON, Capt. P. A.; R. M. Academy.
MCMAHON, Prof., M.A.; Cornell Univ., Ithaca.
MACMURCHY, A., B.A.; Univ. Coll., Toronto.
MADHAVARAO, Prof., M.A.; Vizianagram.
MADISON, ISABEL, B.A.; Cardiff.
MALET, J. C., M.A., F.R.S.; Cork.
MANN, M. F. J., M.A.; Kensington.
MANNHEIM, M.; Prof. à l'Ecole Polytech., Paris.
MARKS, SARAH, B.Sc.; London.
MARTIN, ARTEMAS, M.A., Ph.D.; Washington.
MATHEWS, Professor, B.A.; London.
MATZ, Prof., M.A.; King's Mountain, Carolina.
MERRIFIELD, J., LL.D., F.R.A.S.; Plymouth.
MERRIMAN, MANSFIELD, M.A.; Yale College.
MEYER, MARY S.; Girton College, Cambridge.
MILLER, W. J. C., B.A., (EDITOR); The Paragon, Richmond-on-Thames.
MINCHIN, G. M., M.A.; Prof. in Cooper's Hill Coll.
MITCHESON, T., B.A., L.C.P.; City of London Sch.
MONCK, Prof. H. ST., M.A.; Trin. Coll., Dublin.
MONCOURT, Professor; Paris.
MOON, ROBERT, M.A.; Ex-Fell. Qu. Coll., Camb.
MOORE, H. K., B.A.; Trin. Coll., Dublin.
MOREL, Professor; Paris.
MORGAN, C., B.A.; Instructor to H.M. Fleet.
MORLEY, Prof., M.A.; Haver. Coll., Pennsylvania.
MORRICE, G. G., B.A.; Mecklenburgh Sq., Lond.
MUIR, THOMAS, M.A., F.R.S.E.; Bothwell.
MUKHOPADHYAY, Prof. ASUTOSH, M.A., F.R.S.E.
MUNINDRA NATH RAY, Prof. M.A., LL.B.
NASH, Professor A. M., M.A.; Calcutta.
NEUBERG, Professor; Univ. of Liège.
NEWCOMB, Prof. SIMON, M.A.; Washington.
O'CONNELL, Major-General P.; Bath.
ORCHARD, H. L., M.A., B.Sc.; Hampstead.
OWEN, J. A., B.Sc.; Tennyson St., Liverpool.
PANTON, A. W., M.A.; Fell. of Trin. Coll., Dublin.
PENDLEBURY, C., M.A.; London.
PERRIN, EMILY, B.Sc.; St. John's Wood Park.
PHILLIPS, F. B. W.; Balliol College, Oxford.
PILLAI, C. K., M.A.; Trichy, Madras.
PLAMENEWSKI, H., M.A.; Dahgestan.
POCKLINGTON, H. C., M.A.; Yorks Coll., Leeds.
POLIGNAC, Prince CAMILLE DE; Paris.
POLLEXFEN, H., B.A.; Windermere College.
POOLE, GERTRUDE, B.A.; Cheltenham.
POTTER, J., B.A.; Richmond-on-Thames.
PRESSLAND, A. J., B.A.; Brecon.
PRUDDEN, FRANCES E.; Lockport, New York.
PURSER, Prof. F., M.A.; Queen's College, Belfast.
PUTNAM, K. S., M.A.; Rome, New York.
RADHAKINSHUAN, Prof.; Maharaj. Coll., Madras.
RAWSON, ROBERT; Havant, Hants.
READ, H. J., B.A.; Brasenose Coll., Oxford.
REES, E. W., M.A.; Penarth.
REEVES, G. M., M.A.; Lee, Kent.
REYNOLDS, B., M.A.; Notting Hill, London.
RICHARDS, DAVID, B.A.; Christ's Coll., Brecon.
RICHARDSON, Rev. G., M.A.; Winchester.
ROACH, Rev. T., M.A.; Clifton, Bristol.
ROBERTS, R. A., M.A.; Schol. of Trin. Coll., Dublin
ROBERTS, S., M.A., F.R.S.; London.
ROBERTS, W. R., M.A.; Fell. of Trin. Coll., Dub.
ROBSON, H. C., B.A.; Sidney Sussex Coll., Cam.
ROGERS, L. J., M.A.; Yorkshire Coll., Leeds.
ROY, Prof. KALIPRASANNA, M.A.; Agra.
RUGGERO, SIMONELLI; Università di Roma.
RUSSELL, ALEX., B.A.; Military Coll., Oxford.
RUSSELL, J. W., M.A.; Merton Coll., Oxford.
RUSSELL, R., B.A.; Trinity College, Dublin.
RUTTER, EDWARD; Sunderland.
ST. CLAIR, J. C., M.A.; Ryder Street, London.
SALMON, Professor, D.D., F.R.S., Trin. Coll., Dub.
SANDERSON, Rev. T. J., M.A.; Royston, Cambs.
SARADARANJAN RAY, Prof., M.A.; Dacca.
SARKAR, Prof. BENI MADHAV, M.A.; Agra.
SARKAR, Prof. NILKANTHA, M.A.; Calcutta.
SCHEFFER, Professor; Mercersbury Coll., Pa.
SCHOUTE, Prof. P. H.; University, Groningen.
SCOTT, A. W., M.A.; St. David's Coll., Lampeter.
SCOTT, CHARLOTTE A., D.Sc.; Professor in Bryn Mawr College, Philadelphia.
SCOTT, R. F., M.A.; Fell. St. John's Coll., Camb.
SEGAR, HUGH W.; Trinity Coll., Cambridge.
SEN, RAJ MOHAN; Rajshabye Coll., Bengal.
SEYMOUR, W. R., M.A.; Tunbridge.
SERRET, Professor; Paris.
SHARP, W. J. C., M.A.; Greenwich.
SHARPE, J. W., M.A.; The Charterhouse.
SHARPE, Rev. H. T., M.A.; Cherry Marham.
SHEPHERD, Rev. A. J. P., B.A.; Fell, Q. Coll., Oxf.
SIMMONS, Rev. T. C., M.A.; Grimsby.
SIRCOM, Prof., M.A.; Stonyhurst College.
SIVERLY, WALTER; Oil City, Pennsylvania.
SKRIMSHIRE, Rev. E., M.A.; Llandaff.
SMITH, C., M.A.; Sidney Sussex Coll., Camb.
ST. CLAIR, J. C., M.A.; Ryder Street, London.
STEGGALL, Prof. J. E. A., M.A.; Dundee.
STEEDE, B. H., B.A.; Trin. Coll., Dublin.
STEPHEN, ST. JOHN, B.A.; Caius Coll., Cambridge
STEWART, H., M.A.; Framlingham, Suffolk.
STORR, G. G., B.A.; Clerk of the Medical Council.
SWIFT, C. A., B.A.; Grove House, Walsall.
SWIFT, C. A., B.A.; Southsea, Hants.
SYLVESTER, J. J., D.C.L., F.R.S.; Professor of Mathematics in the University of Oxford, Member of the Institute of France, &c.
SYMONS, E. W., M.A.; Fell. St. John's Coll., Oxon.
TAIT, Prof. P. G., M.A.; Univ., Edinburgh.
TANNER, Prof. H. W. L., M.A.; S. Wales Univ. Coll.
TARLETON, F. A., M.A.; Fell. Trin. Coll., Dub.
TAYLOR, Rev. C., D.D.; Master of St. John's College, Cambridge.
TAYLOR, H. M., M.A.; Fell. Trin. Coll., Camb.
TAYLOR, W. W., M.A.; Ripon Grammar School.
TEBAY, SEPTIMUS, B.A.; Farnworth, Bolton.
TERRY, Rev. T. R., M.A., Ilsley Rectory.
THOMAS, A. E., M.A., Merton College, Oxford.
THOMAS, Rev. D., M.A.; Garsington Rect., Oxford.
THOMPSON, C. H.; St. David's Coll., Lampeter.
THOMSON, Rev. F. D., M.A.; Ex-Fel. St. J. Coll., Cam.
TIRELLI, Dr. FRANCESCO; Univ. di Roma.
TORRY, Rev. A. F., M.A.; St. John's Coll., Camb.
TRAILL, ANTHONY, M.A., M.D.; Fell. T.C.D.
TUCKER, R., M.A.; University Coll., London.
TURRIFF, GEORGE, M.A.; Aberdeen.
VIGARIÉ, EMILE; Laissac, Aveyron.
VINCENZO, JACOBINI; Università di Roma.
VOSE, Professor G. B.; Washington.
WALENN, W. H.; Mem. Phys. Society, London.
WALKER, J. J., M.A., F.R.S.; Hampstead.
WALMSLEY, J., B.A.; Eccles, Manchester.
WARBURTON-WHITE, R., B.A., Salisbury.
WARREN, A. T., M.A.; Brecon.
WARREN, R., M.A.; Trinity College, Dublin.
WATHERSTON, Rev. A. L., M.A.; Bowdon.
WATSON, D., M.A.; Folkestone.
WATSON, Rev. H. W.; Ex-Fell. Trin. Coll., Camb.
WERTSCH, FRANZ; Weimar.
WHALLEY, L. J., B.Sc.; Leytonstone.
WHAPHAM, R. H. W., B.A.; Univ. Coll., Cardiff.
WHITE, J. R., B.A.; Worcester Coll., Oxford.
WHITE, Rev. J., M.A.; Royal Naval School.
WHITESIDE, G., M.A.; Eccleston, Lancashire.
WHITWORTH, Rev. W. A., M.A.; London.
WIENER, L.; Kansas City, Mo., U.S.
WILLIAMS, A. M., M.A.; Aberdeen.
WILLIAMS, C. E., M.A.; Wellington College.
WILLIAMSON, B., M.A.; F. & T. Trin. Coll., Dub.
WILSON, J. M., M.A.; Head-master, Clifton Coll.
WILSON, Rev. J., M.A.; Rect. Bannockburn Acad.
WILSON, Rev. J. R., M.A.; Royston, Cambs.
WOLSTENHOLME, Professor J., M.A., Sc.D.
WOODALL, H. J., B.A.; Stockport.
WOODCOCK, T., B.A.; Twickenham.
WOOLHOUSE, W. S. B., F.R.A.S., &c.; London.
WRIGHT, Dr. S. H., M.A.; Penn Yan, New York.
WRIGHT, W. E., B.A.; Herne Hill.
YOUNG, JOHN, B.A.; Academy, Londonderry.

CONTENTS.

Mathematical Papers, &c.

Solved Questions.

CONTENTS

APPENDIX I.

APPENDIX II.

APPENDIX III.

CORRIGENDUM.

In. Question 9165, on p. 107 of Vol. L., the locus should be $c\tan A.x^2 + a(\tan A + m)xy + (c\tan A + am\tan A - a)y^2 - ac\tan A.x + acy = 0.$

MATHEMATICS

FROM

THE EDUCATIONAL TIMES.

WITH ADDITIONAL PAPERS AND SOLUTIONS.

9830. (Professor CROFTON, F.R.S.)—Prove that (1) the nth power of the series $$S = 1+x+3\frac{x^2}{2!}+4^2\frac{x^3}{3!}+5^3\frac{x^4}{4!}+\dots$$ is $$S^n = 1+n\left\{x+(n+2)\frac{x^2}{2!}+(n+3)^2\frac{x^3}{3!}+(n+4)^3\frac{x^4}{4!}+\dots\right\};$$ and (2) that $\log s = xs$.

9832. (CH. HERMITE, Membre de l'Institut.)—Soit la série
$$y = x+x^2+2x^3+\dots+\frac{(n+2)(n+3)\dots 2n}{2.3\dots n}x^{n+1}+\dots,$$
qui est convergente, si l'on suppose $x < \frac{1}{4}$. On demande de démontrer qu'on a, pour toutes les valeurs de l'exposant w,
$$\frac{1}{(1-y)^w} = 1+wx+\frac{w(w+3)}{1.2}+\frac{w(w+4)(w+5)}{1.2.3}x^3+\dots$$
$$+\frac{w(w+n+1)(w+n+2)\dots(w+2n-1)}{1.2\dots n}x^n+\dots.$$

Solution by Professor SEBASTIAN SIRCOM, M.A.

(9830.) Putting $z = xs$ and reversing the series by the usual method, $x = ze^{-z}$, whence $s = e^{xs}$ and $\log s = xs$. Applying Lagrange's theorem to $z = a+xe^z$, where 0 is to be put for a after the differentiations,
$$z^n = \dots+n\frac{x^n}{n!}\left(\frac{d}{da}\right)^{n-1}(e^{na}a^{n-1})+\dots+n\frac{x^{n+r}}{(n+r)!}\left(\frac{d}{da}\right)^{n+r-1}(e^{(n+r)a}a^{n-1})$$
$$+\dots,$$
$$\left(\frac{d}{da}\right)^{n+r-1}(e^{(n+r)a}a^{n-1}) = e^{(n+r)a}\left(\frac{d}{da}+n+r\right)^{n+r-1}a^{n-1},$$

which, when 0 is put for a, is

$$(n+r-1)\frac{n+r-2}{2}\cdots\frac{n}{r}(n+r)^r(n-1)! = \frac{(n+r)!}{r!}(n+r)^{r-1}.$$

The first term that does not vanish is x^n, and we have

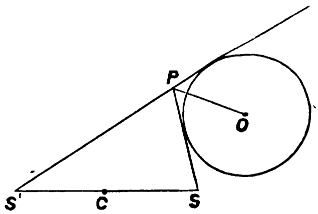
$$z^n = x^n + n\left\{x^{n+1} + (n+2)\frac{x^{n+2}}{2!} + (n+3)^2\frac{x^{n+3}}{3!} + (n+4)^3\frac{x^{n+4}}{4!} + \dots\right\},$$

which gives the required result.

(9832.) This question can be treated by the same method.

9855. (The Editor.)—Find the locus of the intersections of tangents drawn from two fixed points to a variable circle around another fixed point.

Solution by Professor Wolstenholme; W. S. Foster, M.A.; *and others.*



Let S, S′ be the two fixed points, O the centre of the variable circle, P a point of intersection of tangents to such a circle from S, S′. Then, since OP bisects one of the angles between PS, PS′, P is a point of contact of a tangent drawn from O to some conic having S, S′ for foci, and the locus is the well-known circular nodal cubic whose equation is

$$(x^2+y^2-x\mathrm{X}-y\mathrm{Y})(\mathrm{X}y-\mathrm{Y}x) = c^2(x-\mathrm{X})(y-\mathrm{Y}) \quad\text{..............(A)},$$

the origin C bisecting S′S, along which is the axis of x, S′C = CS = c, (X, Y) the coordinates of O. It is well known that the envelope of the normal at P to the confocal to which OP is the tangent at P (also the envelope of the polar of O) is a parabola of which CO is directrix, and which touches the axes of coordinates, and the nodal tangents to the cubic at O. This proves that the locus of P is a pedal of this parabola with respect to a point on the directrix, and hence that this parabola and the cubic have triple three-point contact at the feet of the normals from O to the parabola. Only one of these is real of course. This is a general property of any nodal cubic and the conic which I have called the companion conic, but I do not remember to have seen this particular case of it mentioned, and I should like to see a neat algebraical proof that the two curves (A), and $(x\mathrm{X}-y\mathrm{Y}-c^2)^2+4\mathrm{XY}xy=0$ ....(B), have triple two-point contact.

It is obvious that P is also the locus of the intersections of common tangents to *any* conic whose foci are S, S′, and *any* circle whose centre is O, the locus being definite, although two parameters appear to be involved. The following properties of this cubic may have some interest.

(1) If OP, OQ be tangents from O to any conic whose foci are S, S′, and U the point of contact of PQ with (B), then, if OR be perpendicular on PQ, [UPQR] = − 1 ; (2) the tangents to (A) at P, Q intersect in a point R′ lying upon (A) ; and the tangents to (A) at R, R′ also intersect in a point upon (A) ; (3) if OK be a normal drawn from O to (A), the tangent at K will pass through an inflexion of (A) ; (4) the positions of these inflexions are given by the equation $\tan^3\theta = \tan\alpha$, where α is the angle which CO makes with the straight line bisecting SOS′, and θ is the angle which the straight line from O to an inflexion makes with the same bisector; (5) the perpendiculars from O upon the tangents to (A) at P, Q are equal, &c., &c.

Referred to polar coordinates, the equation may be written

$$r = a\sin\theta\cos\theta \,/\, \sin(\alpha-\theta),$$

where O is pole, the prime radius bisects the angle SOS′, α is the angle which OC makes with the prime radius, and $a = \mathrm{OS}\,.\,\mathrm{OS}'/\mathrm{OC}$.

This is one of the exceptional cases where a locus is determinate, although the point appears to depend upon two parameters.

Query.—If α, β be two independent parameters, and the coordinates of a point be given by $x = f_1(\alpha, \beta)$, $y = f_2(\alpha, \beta)$, under what circumstances will the resultant of these equations with respect to α be independent of β? Of course, in any such case, it must be possible to find two equations $x = \phi_1(\lambda)$, $y = \phi_2(\lambda)$, where λ is some function of α, β; exemplify this in the present case.

[Let A be the centre of the circles, and B, C the fixed points; ABC the triangle of reference; h the perpendicular from A on BC; and $\alpha - \mu\beta = 0$, $\alpha - \nu\gamma = 0$ the equations to the tangents from B, C; then the perpendiculars from A on these lines are equal; therefore

$$h/(1+\mu^2-2\mu\cos C)^{\frac{1}{2}} = h/(1+\nu^2-2\nu\cos B)^{\frac{1}{2}},$$

and $\mu^2-2\mu\cos C = \nu^2-2\nu\cos B$; thus $\alpha(\beta^2-\gamma^2) = 2\beta\gamma(\gamma\cos C-\beta\cos B)$ is the locus of the intersections.]

9904. (Professor DE LONGCHAMPS.)—Si l'on a $\tan\alpha = m\,.\,i$, on a $\tan\frac{1}{3}\alpha = i\,(m+u+v)$, après avoir posé

$$u = \left\{(m+1)^2(m-1)\right\}^{\frac{1}{3}}, \quad v = \left\{(m-1)^2(m+1)\right\}^{\frac{1}{3}}.$$

Montrer comment la Question 9848 conduit à cette conclusion.

Solution by Professor WOLSTENHOLME, M.A., Sc.D.

Put $\tan\frac{1}{3}\alpha = xi \left\{= (y+m)\,i\right\}$, when $\tan\alpha = \dfrac{3xi+ix^3}{1+3x^2} = mi$,

or $\quad x^3-3mx^2+3x-m=0$, or $y^3-3y\,(m^2-1)-2m\,(m^2-1) = 0$,

and the solution of this is, by Tartaglia's rule, $y = u+v$, where

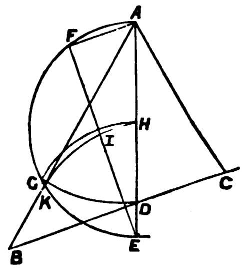
$$uv = m^2-1, \quad u^3+v^3 = 2m\,(m^2-1)\,;$$

i.e., $\quad u^3 = (m+1)^2(m-1), \quad v^3 = (m-1)^2(m+1),$

whence the result follows.

4146. (Professor Evans, M.A.)—Construct a triangle, being given the product of two sides, the medial line to the third side, and the difference of the angles adjacent to the third side.

Solution by D. Biddle.



Let AD = the medial line, AE = the product AB . AC, and AEF = half the difference of the angles B and C. Upon AE, regarded as unity, describe the semi-circle, cutting EF in F. With centre A, radius AD, describe an arc, cutting the semi-circle in G, and with E as centre, and EG as radius, describe the arc GH, cutting AE in H. Draw HI parallel to AF, and with centre E radius EI describe the arc IK, cutting the semi-circle in K. Join AK and produce; also make

$$\angle \mathrm{EAC} = \mathrm{EAK},$$

and through D draw BC at right angles with EF . ABC is the triangle required.

For $$2\mathrm{AEF} = \mathrm{ADB} - \mathrm{ADC} = \mathrm{ACB} - \mathrm{ABC}.$$

Moreover, $\sin \mathrm{ADB} : \mathrm{AB} = \sin \mathrm{B} : \mathrm{AD}$, and $\sin \mathrm{ADC} : \mathrm{AC} = \sin \mathrm{C} : \mathrm{AD}$

whence $$\mathrm{AB}.\mathrm{AC} = \frac{\mathrm{AD}^2 \sin \mathrm{ADB} \sin \mathrm{ADC}}{\sin \mathrm{B} \sin \mathrm{C}} = \frac{\mathrm{AD}^2 \cos^2 \mathrm{AEF}}{\cos^2 \mathrm{AEF} - \sin^2 \mathrm{EAB}},$$

and $$\sin \mathrm{EAB} = \left(1 - \frac{\mathrm{AD}^2}{\mathrm{AB}.\mathrm{AC}}\right)^{\frac{1}{2}} \cos \mathrm{AEF}.$$

Now, in the construction, AB . AC = unity = AE, and

$$\mathrm{EH} = \mathrm{EG} = (1 - \mathrm{AD}^2)^{\frac{1}{2}}.$$

Also, $$\mathrm{EK} = \sin \mathrm{EAB} = \mathrm{EI} = \mathrm{EH}.\mathrm{EF} = (1 - \mathrm{AD}^2)^{\frac{1}{2}} \cos \mathrm{AEF}.$$

9834. (Professor Reuchle.)—Les droites qui joignent les sommets du triangle ABC aux points de contact des côtés opposés avec le cercle inscrit I, se rencontrent en un point Γ (point de Gergonne) ; celles qui joignent les milieux des côtés de ABC aux centres I_a, I_b, I_c des cercles exinscrits correspondants se rencontrent en un point U. Démontrer que la droite ΓU passe par le centre de gravité de ABC.

Solution by R. Tucker, M.A.; G. G. Storr, M.A.; *and others.*

The Gergonne-point (Γ) is given by

$$a\alpha(s-a) = b\beta(s-b) = c\gamma(s-c) \qquad\qquad (1),$$

and U by $\quad (b-c)\alpha + b\beta - c\gamma = 0, \quad -a\alpha + (c-a)\beta + c\gamma = 0,$

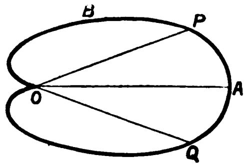
therefore it is given by $\alpha/(s-a) = \beta/(s-b) = \gamma/(s-c)$ ....(2).
The equation to ΓU is readily found to be

$$a\alpha(b-c)(s-a) + b\beta(c-a)(s-b) + c\gamma(a-b)(s-c) = 0,$$

which evidently passes through the centroid.

9724. (W. J. Greenstreet, M.A.)—O is the pole of the cardioid, $r = a(1+\cos\theta)$; OP, OQ trisect the area of the cardioid; and the angle POQ is denoted by 2ϕ. Prove that $\sin\phi(4+\cos\phi) = \pi - 3\phi$.

Solution by Rev. T. Galliers, M.A.; and G. G. Storr, M.A.



Since OP, OQ, trisect the area of the cardioid, area AOP $= \frac{1}{3}$APBO.

Now area APBO $= \frac{1}{2}\int_0^\pi r^2\,d\theta$

$$= \tfrac{1}{2}a^2\int_0^\pi (1+\cos\theta)^2\,d\theta = \tfrac{3}{4}\pi a^2,$$

and area AOP $= \frac{1}{2}a^2\left\{\frac{3}{2}\phi + 2\sin\phi + \frac{1}{2}\sin\phi\cos\phi\right\}$;

therefore $3\phi + 4\sin\phi + \sin\phi\cos\phi = \pi$, or $\sin\phi(4+\cos\phi) = \pi - 3\phi$.

9586. (Professor Chakravarti, M.A.)—If the sum of the axes of an ellipse be a constant (s), show that its average area is $\frac{1}{24}\pi s^2$.

Solution by Artemas Martin, LL.D.

Let x and $s-x$ be the axes, then the area of the ellipse is $\frac{1}{4}\pi x(s-x)$, and the average area required is

$$\int_{\frac{1}{2}s}^{s} \tfrac{1}{4}\pi x(s-x)\,dx \div \int_{\frac{1}{2}s}^{s} dx = \frac{\pi}{2s}\int_{\frac{1}{2}s}^{s} x(s-x)\,dx = \tfrac{1}{24}\pi s^2.$$

9768. (E. Lemoine.)—Soient ABC, $A_1B_1C_1$ deux triangles. Démontrer que le lieu des points M et le lieu des points M_1, tels que AM, BM, CM soient respectivement parallèles à A_1M_1, B_1M_1, C_1M_1, sont des coniques. Le lieu de M est une conique circonscrite à ABC, celui de M_1 une autre conique circonscrite à $A_1B_1C_1$. Examiner les cas particuliers, où les côtés de $A_1B_1C_1$ sont parallèles aux hauteurs, aux bissectrices, aux médianes, aux symédianes, aux antiparallèles, du triangle ABC.

Solution by Professor Schoute.

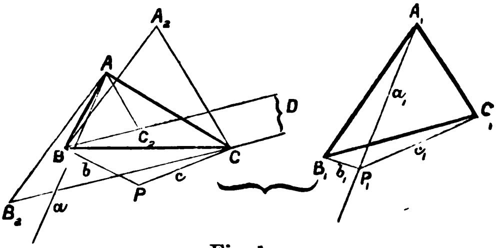


Fig. 1.

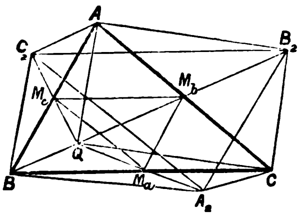


Fig. 2.

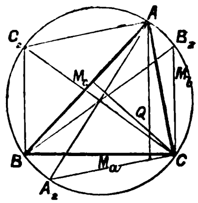


Fig. 3.

Considérons deux couples de parallèles, b et b_1 par B et B_1, c et c_1 par C et C_1 (Fig. 1). Soit P le point d'intersection de b et c, P_1 celui de b_1 et c_1. Si P_1 parcourt la droite a_1 par A_1, les deux faisceaux (b_1) et (c_1) sont liés perspectivement à l'aide de la ponctuelle (P_1) sur a_1. Donc les faisceaux (b) et (c) des droites parallèles b et c sont liés projectivement, de manière que P parcourt une conique C^2, quand P_1 décrit a_1. Quelle que soit la direction de la droite a_1 par A_1 la conique C^2 passe par quatre points, les points de base B et C, le point infiniment éloigné D de B_1C_1 (correspondant au point P_1 commun à a_1 et B_1C_1) et le point A_2 (correspondant au point A_1). Donc les coniques C^2, qui correspondent aux droites a_1 par A_1, forment un faisceau (C^2) aux quatre points de base A_2, B, C, D. Ce faisceau est rapporté projectivement au faisceau (a_1), p.e., à l'aide des points mobiles des éléments correspondants situés à l'infini. Il est donc lié projectivement au faisceau (a) des droites a parallèles à a_1. Et comme il est évident que le point P est un point M quand P se trouve sur la droite correspondante a, le lieu des points M est le lieu des points d'intersection des éléments correspondants des deux faisceaux (C^2) et (a). Ce lieu est une cubique par A_2, B, C, D, et A, dont se détache la droite à l'infini, c.-à-d. la conique circonscrite à ABC, qui passe par A_2 et par les points B_2 et C_2 obtenus d'une manière analogue. Et la conique circonscrite à $A_1B_1C_1$, qui est le lieu de M_1, s'obtient par la même construction après l'échange des deux triangles donnés.

Ce que nous venons de prouver, démontre le théorème suivant. Quand sur les côtés BC, CA, AB d'un triangle ABC on décrit des triangles BCA_2, CAB_2, ABC_2, dont les côtés nouveaux sont parallèles aux côtés homologues d'un autre triangle $A_1B_1C_1$ situé dans le plan de ABC, les six

points A, B, C et A_2, B_2, C_2 sont sur une conique. D'ailleurs cela est évident d'après le théorème de Pascal par rapport à l'hexagone $AB_2CA_2BC_2$ à trois couples de côtés opposés parallèles.

Un cas rémarquable se présente, quand les droites par A, B, C respectivement parallèles à B_1C_1, C_1A_1, A_1B_1 concourent en un même point Q (Fig. 2); alors on peut faire coïncider le triangle $A_2B_2C_2$ avec le triangle ABC à l'aide d'une rotation autour d'un axe normal au plan des triangles. Le point où cet axe perce le plan est le centre de la courbe. L'hexagone $AB_2CA_2BC_2$ se compose de trois parallélogrammes. Le rôle du point Q dans le triangle $M_aM_bM_c$ des milieux des côtés de ABC est aussi le rôle de ce point dans le triangle $A_2B_2C_2$, etc.

Si Q est le point de concours des hauteurs, la conique est un cercle (Fig. 3); si Q est le centre de gravité, Q est en même temps le centre de la conique, qui touche en A, B, C les parallèles à BC, CA, AB, de manière que ABC est un triangle d'aire maximum dans la conique, etc. Si Q est le centre de gravité du périmètre de ABC, il est le centre du cercle inscrit de $A_2B_2C_2$.

Quand y_1, y_2, y_3 sont les coordonnées normales de Q par rapport à ABC et z_1, z_2, z_3 par rapport à $M_aM_bM_c$, l'équation de la conique C^2 est

$$\sum_1^3{}_i \frac{a_i y_i z_i}{x_i} = 0,$$

où a_1, a_2, a_3 représentent les côtés de ABC.

9753. (Professor BEYENS.)—Mener une tangente à une circonférence qui passe par le point du rencontre inaccessible de deux droites données.

Solution by J. C. St. CLAIR; SARAH MARKS, B.Sc.; *and others.*

Let P, P′ be the polars of the given lines L, L′. The tangents at the points where the line PP′ meets the circle pass through the intersection of L, L′.

1228. (N'IMPORTE.)—A messenger M starts from A towards B (distance a) at a rate of v miles per hour; but, before he arrives at B, a shower of rain commences at A and at all places occupying a certain distance z towards, but not reaching beyond, B, and moves at the rate of u miles an hour towards A; if M be caught in this shower, he will be obliged to stop until it is over; he is also to receive for his errand a number of shillings inversely proportional to the time occupied in it, at the rate of n shillings for one hour. Supposing the distance z to be unknown, as also the time at which the shower commenced, but all events to be equally probable, show that the value V of M's expectation is, in shillings,

$$V = \frac{nv}{a}\left\{\frac{1}{2} - \frac{u}{v} + \frac{u(u+v)}{v^2}\log\frac{u+v}{u}\right\}.$$

Solution by W. S. FOSTER.

Let x = distance M has gone when the shower begins; then the time he takes getting to B $= \frac{a}{v} + \frac{z-x}{u}$, if $z > x$, and $= \frac{a}{v}$ if $z < x$; thus

$$V = n\int_0^a \left\{\int_0^x \frac{v\,dz}{a} + \int_x^a \frac{uv\,dz}{au+v(z-x)}\right\} dx \Big/ \int_0^a\int_0^a dx\,dz$$

$$= \frac{n}{a^2}\int_0^a \left\{\frac{vx}{a} + u\log\frac{a(u+v)-vx}{au}\right\} dx$$

$$= \frac{n}{a^2}\left\{\frac{va}{2} - au\log au + u\int_0^a \log\left[a(u+v)-vx\right] dx\right\}$$

$$= \frac{n}{a^2}\left\{\frac{av}{2} - au + a(u+v)\frac{u}{v}\log\frac{u+v}{u}\right\}$$

$$= \frac{nv}{a}\left\{\frac{1}{2} - \frac{u}{v} + \frac{(u+v).u}{v^2}\log\frac{u+v}{u}\right\}.$$

8342. (BELLE EASTON.) — An arithmetical, geometrical, and harmonical progression have each the same number of terms, and the same first and last terms, a and l respectively; the sums of all the terms of the three series respectively are s_1, s_2, s_3, and their continued products are p_1, p_2, p_3; show that, when the number of terms is indefinitely increased,

$$\frac{s_1}{s_2} = \frac{l+a}{2(l-a)}\log_e\left(\frac{l}{a}\right),\quad \frac{s_1^2}{s_2 s_3} = \frac{(a+l)^2}{4al},\quad \text{and}\quad \frac{p_1 p_3}{p_2^2} = 1.$$

Solution by G. G. STORR, M.A., *and* Rev. T. GALLIERS, M.A.

We have $s_1 : s_2 = \frac{1}{2}n(a+l) : a\left\{\left(\frac{l}{a}\right)^{1/(1-1/n)} - 1\right\} \Big/ \left\{\left(\frac{l}{a}\right)^{1/(n-1)} - 1\right\}$

$$= \frac{l+a}{2(l-a)}\log_e\left(\frac{l}{a}\right),\quad \text{for } n = \infty,$$

$$\because\ \mathrm{L}^t_{n=\infty}\, n\left\{\left(\frac{l}{a}\right)^{1/(n-1)} - 1\right\} = \log_e\left(\frac{l}{a}\right).$$

Again, the r^{th} term of the H. P. $= la \Big/ \left\{l - \frac{r-1}{n-1}(l-a)\right\}$

therefore $\frac{s_3}{s_1} = \frac{2}{n(l+a)}\Sigma_{r=0}^{r=\infty}\ la \Big/ \left\{l - \frac{r-1}{n-1}(l-a)\right\}$

$$= \frac{2la}{n(l^2-a^2)}\int_0^1 dx \Big/ \left\{\frac{l}{l-a} - x\right\} = \frac{2la}{l^2-a^2}\log_e\left(\frac{l}{a}\right);$$

therefore $$\frac{s_1^2}{s_2 s_3} = \frac{(a+l)^2}{4al}.$$

Lastly, the product of the $(n-r+1)^{\text{th}}$ term of the A. P., and the r^{th} term of the H. P. $= la$; hence $p_1 p_3 = (la)^n$. But if r be the common ratio of the G. P.,

$$p_2 = a \,.\, ar \,.\, ar^2 \ldots ar^{n-1} = a^n r^{\frac{1}{2}n(n-1)} = a^n (l/a)^{\frac{1}{2}n}, \text{ thus } p_1 p_3 p_2^{-2} = 1.$$

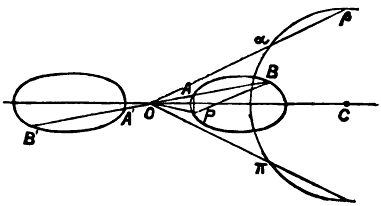
9846. (Professor Genese, M.A.)—Prove that any fixed diameter of an oval of Cassini determines two chords which subtend angles whose difference is constant at any point of the oval.

Solution by W. J. Johnston, M.A.; Emily Perrin; *and others.*

I shall assume the following theorem of M. Laisant, viz., If O is a fixed point, and the points π, P such that we have the equipollence

$$OP = (O\pi)^{\frac{1}{2}},$$

then, if the locus of π is a circle $\alpha\beta\pi$ whose centre is C, that of P is an oval of Cassini with centre O and axis OC.



Let OAB be the fixed diameter of the oval, and let the points α, β, π correspond to A, B, P, so that $O\alpha = OA^2 = OA'^2$, &c. Then, using equipollences,

$$PA \,.\, PA' = (OA - OP)(-OA - OP) = OP^2 - OA^2 = O\pi - O\alpha = \alpha\pi;$$

similarly, $\quad PB \,.\, PB' = \beta\pi;$

therefore $\quad PA \,.\, PA'/PB \,.\, PB' = \alpha\pi/\beta\pi.$

Hence, noting the inclinations, we have the theorem enunciated, since the angle $\alpha\pi\beta$ is constant.

9575. (J. C. Malet, F.R.S.)—If the plane of a triangle ABC cut three spheres S_1, S_2, S_3 at equal angles, and if through AB a pair of tangent planes be drawn to S_3, through BC a pair to S_1, and through AC a pair to S_2, prove that the six tangent planes so drawn touch the same sphere.

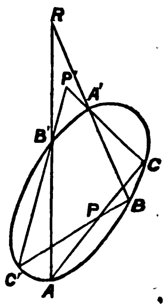
Solution by W. S. M'Cay, M.A.

The plane of the triangle ABC passes through an axis of similitude of S_1, S_2, S_3. Hence (denoting for shortness the positions of the sides of the triangle by a, b, c) we can find a line b', in the plane, the homologue to S_1 of b to S_3 with respect to that centre of similitude which lies in the plane, and a line c' the homologue to S_1 of c to S_3 with respect to their centre of similitude in the plane.

The tangent planes through a to S_1 are two of those in question, and the tangent planes through b' and c' to S_1 are parallel to the other two pair in question; and, since b and b' are parallel and also c and c', the six planes through a, b, c are homothetic with those through a, b', c', and therefore touch a sphere, since the others touch S_1.

9661. (Professor Wolstenholme, M.A., Sc.D.)—On a conic are taken any six points A, B, C, A', B', C'; AC, BC' meet in P; A'C, B'C' in P. Prove that PP', AB', A'B concur in one point. (If AC, B'C' meet in Q' A'C, BC' in Q', it is clear that QQ', AB, A'B' also meet in a point).

Solution by Professors Madhavarao, Beyens, *and others.*



Let B'C' = 0 denote the equation of the line joining B', C'. Then, since the conic circumscribes the quadrilateral B'C'BA', its equation may be expressed by B'C'. A'B − BC'. A'B' = 0. Since it also circumscribes B'ACA', the same conic may be represented by AB'. A'C − AC. A'B' = 0. Subtracting, we have

$$B'C'.\,A'B - AB'.\,A'C = A'B'\,(BC' - AC),$$

therefore B'C'. A'B − AB'. A'C, which represents a figure circumscribing the quadrilateral formed by B'C', A'C, A'B, AB' (*i.e.*, the figure B'P'A'R), being resolvable into two factors, represents the two diagonals. Therefore A'B' (BC' − AC) represents the diagonals A'B' and P'R. Therefore BC' − AC represents the line P'R, and it also passes through P. To show that QQ', AB, A'B' meet in a point, we get for the equation of the conic A'B'. BC' = B'C'. A'B or A'C . AB = AC . A'B, from which A'B'. BC' − A'C . AB = (B'C' − AC) A'B, whence the required result follows.

9671. (Professor Neuberg.)—On donne, dans un même plan, un triangle ABC et une circonférence Δ. D'un point quelconque M de Δ, on abaisse les perpendiculaires MA', MB', MC' sur les côtés de ABC, et l'on construit le triangle A'B'C'. Sur une base fixe $\alpha\beta$, on construit un triangle $\alpha\beta\gamma$ semblable au triangle A'B'C'. Démontrer que, lorsque M décrit la circonférence Δ, le point γ décrit une seconde circonférence Δ'.

Solution by the Proposer.

Soumettons la figure à une transformation par rayons vecteurs réciproques en prenant pour pôle d'inversion le sommet C. Soient A_1, B_1, M_1 les inverses des points A, B, M. Les triangles semblables CMA et CA_1M,

CMB et CB_1M_1, CA_1B_1 et CBA donnent, P étant la puissance,

$$\frac{A_1M_1}{AM} = \frac{CM_1}{CA} = \frac{CM_1 \,.\, CM}{CA \,.\, CM} = \frac{P}{CA \,.\, CM'},$$

$$\frac{B_1M_1}{BM} = \frac{P}{CB \,.\, CM'} \qquad \frac{A_1B_1}{AB} = \frac{P}{CA \,.\, CB'}$$

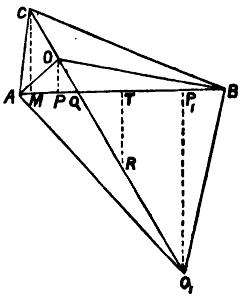
d'où $\quad A_1M_1 : B_1M_1 : A_1B_1 = AM \,.\, CB : BM \,.\, CA : AB \,.\, CM'$

$= AM \sin A : BM \sin B : CM \,.\, \sin C = B'C' : C'A' : A'B'.$

Donc le triangle $B_1A_1M_1$ est toujours semblable au triangle $A'B'C'$. Or, la base A_1B_1 est fixe, et les points M, M_1 décrivent deux lignes inverses ; lorsque M décrit une circonférence Δ, M_1 décrit aussi une circonférence Δ' ou une droite. Le théorème est donc démontré, les points B_1 et A_1 étant les points α, β de l'énoncé.

9673. (Professor BORDAGE.) — Construct a triangle, knowing the centre O of the in-circle, the mid-point T of a side AB, and the point M where the perpendicular CM cuts AB.

Solution by Professor SCHOUTE ; D. BIDDLE ; *and others.*



The centres O, O_1 of the in-circle and the ex-circle to AB are divided harmonically by C and Q ; hence, when R is the mid-point of OO_1, we have $OR^2 = CR \,.\, QR$.

Now R is the centre of the circle AOBO, and its projection on AB is T. By projection on AB, we find $\quad PT^2 = MT \,.\, QT$;

hence QT is to be a third proportional to MT and PT, etc.

9677. (J. C. MALET, F.R.S.)—If the modulus (c) and the amplitude (ϕ) of the elliptic integral F (c, ϕ), be given by the equations

$$c = \cos \tfrac{1}{12}\pi, \quad \cos\phi = 2 - \sqrt{3}, \quad \text{then } F(c, \phi) = \left\{ \sqrt{\pi}\, \Gamma(\tfrac{1}{3}) \right\} / \left\{ 3^{\frac{3}{4}} \,.\, \Gamma(\tfrac{5}{6}) \right\}.$$

Solution by Professor SEBASTIAN SIRCOM, M.A.

Since $c = \dfrac{1 + \sqrt{3}}{2\sqrt{2}}$, we have (ROBERTS' *Integral Calculus*, p. 133)

$$\int \frac{d\phi}{(1 - c^2 \sin^2 \phi)^{\frac{1}{2}}} = -3^{\frac{1}{4}} \int \frac{dx}{(1 - x^3)^{\frac{2}{3}}},$$

where $x = 1 - 3^{\frac12}\tan^2\frac12\phi$. For the limits, we have $x = 1$ when $\phi = 0$, and $x = 0$ when $\cos\phi = 2-3^{\frac12}$, hence $F(c, \phi) = 3^{\frac14}\int_0^1 \frac{dx}{(1-x^3)^{\frac12}}$; writing x for x^3, this becomes $\frac{1}{3^{\frac34}}\int_0^1 x^{\frac13-1}(1-x)^{\frac12-1}\,dx$, equal to the required result.

9422. (Professor BORDAGE.) — If A, B, C be three given points ($BC = a$, $CA = b$, $AB = c$), α, β, γ their distances from a fixed straight line, and Δ the area of the triangle ABC, prove that

$$a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 - (a^2+b^2-c^2)\,\alpha\beta - (b^2+c^2-a^2)\,\beta\gamma - (c^2+a^2-b^2)\,\gamma\alpha = 4\Delta^2.$$

Solution by J. McMAHON, B.A.; R. KNOWLES, B.A.; *and others.*

Let $lx + my + nz = 0$ be the equation to the given line; then

$$\alpha = \frac{2\Delta \, . \, l}{a} \Big/ (l^2 + m^2 + n^2 - 2mx\cos A - 2nl\cos B - 2lm\cos C)^{\frac12}, \text{ \&c.}$$

Substituting these values of α, β, γ in the sinister, we get the result.

9048. (ASPARAGUS.)—A triangle ABC is inscribed in a circle, the symmedian through A meets the circle again in D, and the tangent at A meets BC in A′; through A′ is drawn any straight line meeting the circle in P, Q; prove that a conic can be drawn touching AB, AC in B, C, and touching DP, DQ in P, Q. Also generalise the theorem by projection, remembering that the symmedian at A passes through the intersection of tangents at B, C.

[The generalised theorem is as follows:—A triangle ABC is inscribed in a conic, AD is a chord of the conic passing through the pole of BC, and A′ is the point where the tangent at A meets BC; any straight line through A′ meets the conic in P, Q; a conic can be drawn touching AB, AC in B, C, and touching DP, DQ in P, Q.]

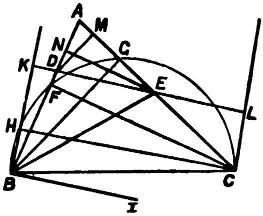
Solution of the generalised theorem by the PROPOSER.

$lyz + mzx + nxy = 0$ the conic; at A′, $x = 0$, $ny + mz = 0$; at D, $-2x/l = y/m = z/n$, $\lambda x + ny + mz = 0$ equation of PQ; then the conic $\lambda x^2 = lyz$ [*i.e.*, $x(\lambda x + ny + mz) = lyz + mzx + nxy$] touches AB, AC at B, C, and passes through P, Q; and the pole of PQ with respect to it is (XYZ), where $2\lambda xX = l(yZ + zY)$ coincides with $\lambda x + ny + mz = 0$, or $-2X/l = Y/m = Z/n$; so that D is the pole of PQ and the conic touches DP, DQ at P, Q. [The theorem becomes obvious, also, by taking B, C as the isotropic points.]

9367. (F. MORLEY, M.A.)—In the sides AB, AC of a triangle ABC, find points D, E, such that BD = DE = EC.

4130. (The EDITOR.)—Given the vertical angle, the difference of the sides, and the sum of the base and less side, construct the triangle.

Solution by D. BIDDLE.



(9367.) Upon BC describe the semicircle BFGC, and make BH = CF - BG. Join CH, and draw BI parallel to CH. Bisect the angle ABI by BE, and draw DE parallel to CH. D and E are the points required. For, analytically, if DE = BD, then EN = BK, and if EC = DE, then DM = CL. Moreover, if EC = BD, then we have EN − DM = CF − BG = BH. But DE = BD by construction, since

$$EBI = DEB = DBE;$$

and, since BK, CL are identical with what they would be under the required conditions, it is impossible but that EC = DE = BD.

(4130). If we regard AB = the given sum (of the base and the less side), and AC = AB + the given difference (of the sides), it is evident that ADE is the triangle required, and its construction that above given.

[For Solutions of a generalised form of 9367, see Vol. XLII., p. 64, and Vol. XLIX., p. 51.]

9875. (E. W. REES, B.A.)—If O be the orthocentre of a triangle ABC, whose pedal triangle is DEF, and if σ_1, σ_2, σ_3 are the symmedian points of the triangles OBC, OCA, OAB respectively; prove that (1) the trilinear coordinates of σ_1, σ_2, σ_3 are

$$\frac{\alpha}{a} = \frac{\beta\cos B}{b\cos(C-A)} = \frac{\gamma\cos C}{c\cos(A-B)},\quad \frac{\alpha\cos A}{a\cos(B-C)} = \frac{\beta}{b} = \frac{\gamma\cos C}{c\cos(A-B)},$$

$$\frac{\alpha\cos A}{a\cos(B-C)} = \frac{\beta\cos B}{b\cos(C-A)} = \frac{\gamma}{c};$$

(2) $A\sigma_1$, $B\sigma_2$, $C\sigma_3$ intersect in

$$\frac{\alpha\cos A}{a\cos(B-C)} = \frac{\beta\cos B}{b\cos(C-A)} = \frac{\gamma\cos C}{c\cos(A-B)};$$

and (3) this is the symmedian point of the triangle DEF.

Solution by R. TUCKER, M.A.; R. KNOWLES, B.A.; *and others.*

The coordinates of mid-point (L) of OB, and mid-point (L′) of CE are proportional to

[cos B cos C, cos (C − A), cos A cos B], [cos C sin C, 0, sin B + sin C cos A];

therefore equations of LL′ and MM′ (M on OC) are

$$\alpha(b+c\cos A)\cos(C-A)-\beta b\cos B\cos C-c\gamma\cos C\cos(C-A) = 0,$$
$$\alpha(c+b\cos A)\cos(A-B)-\beta b\cos B\cos(A-B)-c\gamma\cos B\cos C = 0;$$

whence coordinates of σ_1 are

$$\frac{\alpha}{a}=\frac{\beta\cos B}{b\cos(C-A)}=\frac{\gamma\cos C}{c\cos(A-B)};$$

those of σ_2, σ_3, and the result of (2), follow at once.

Let the symmedian through D cut FE in X, then

$$FX : EX = FD^2 : DE^2 = b^2\cos^2 B : c^2\cos^2 C.$$

It will be found that the coordinates of X are proportional to

$$2\sin A\cos B\cos C\cos(B-C),\quad \cos A\cos C\sin 2C,\quad \cos A\cos B\sin 2B,$$

and those of D to 0, cos C, cos B; hence equation to DX is

$$-\cos^2 A\sin(B-C)\,\alpha+\sin A\cos(B-C)(\beta\cos B-\gamma\cos C) = 0.$$

It is easily verified that the values of (2) satisfy this equation.

The equation to the line through the two S-points is

$$\alpha\cos^2 A\sin(B-C)+\ldots+\ldots = 0,$$

which is the equation to KO. This result was first given, it is believed, by M. E. VAN AUBEL.

It is interesting at the present stage to have the coordinates of new points put on record. Let the tangents to the N. P. circle at E, F intersect in A′; this point is determined by

$$\alpha\cos A/\sin A\cos(B-C) = -\beta/\sin(C-A) = \gamma/\sin(A-B),$$

similarly B′, C′ by

$$\alpha/\sin(B-C) = \beta\cos B/\sin B\cos(C-A) = -\gamma/\sin(A-B),$$
$$-\alpha/\sin(B-C) = \beta/\sin(C-A) = \gamma\cos C/\sin C\cos(A-B).$$

Then equations to DA′, EB′, FC′, are

$$\alpha\cos^2 A\sin(B-C)-\beta\sin A\cos B\cos(B-C)+\gamma\cos C\sin A\cos(B-C)=0,$$
$$\alpha\cos A\sin B\cos(C-A)+\beta\cos^2 B\sin(C-A)-\gamma\sin B\cos C\cos(C-A)=0,$$
$$-\alpha\sin C\cos A\cos(A-B)+\beta\cos B\sin C\cos(A-B)+\gamma\cos^2 C\sin(A-B) = 0.$$

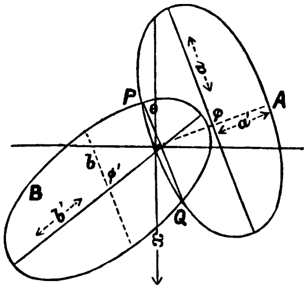
These intersect in the S.-point of DEF, and give for it

$$\alpha\cos A/a\cos(B-C) = \beta\cos B/b\cos(C-A) = \gamma\cos C/c\cos(A-B).$$

8934. (ALICE GORDON, B.Sc.,)—Two ellipses A and B in a plane intersect (in two points) along PQ. The diameters parallel to PQ in A and B are a and b, and the conjugates a' and b'; a homogeneous strain acts on them (in their plane) twisting them into two intersecting ellipses, having one of their equiconjugates parallel to PQ. Find (1) the ratio of the elongations along the principal axes of strain and their inclination to PQ; also (2) the conditions for the strained ellipses becoming circles.

Solution by the PROPOSER.

Let θ be the inclination of one of the principal axes of strain to PQ (viz. x). The new conjugates in each ellipse will be the displaced positions of the old conjugates a', b'. Let λ be the elongation along this axis (x), λz that along the perpendicular axis. Then the new a will be



$$= \lambda a \sqrt{\cos^2\theta + z^2 \sin^2\theta},$$

and also by hypothesis

$$= \text{new } a'$$

$$= \lambda a' \sqrt{\cos^2(\theta+\phi) + z^2 \sin^2(\theta+\phi)},$$

$$\therefore \quad \frac{a^2}{a'^2} = \frac{\cos^2(\theta+\phi) + z^2 \sin^2(\theta+\phi)}{\cos^2\theta + z^2\sin^2\theta}.$$

Let $$z^2 = 1 + x^2,$$

then we have $$a^2 - a'^2 = x^2 \left\{a'^2 \sin^2(\theta+\phi) - a^2 \sin^2\theta\right\};$$

similarly, $$b^2 - b'^2 = x^2 \left\{b'^2 \sin^2(\theta+\phi') - b^2 \sin^2\theta\right\},$$

from which x and θ can be determined; in fact, the following equation gives $\cot\theta$,

$$\frac{a^2 - a'^2}{b^2 - b'^2} = \frac{a'^2(\cos\phi + \sin\phi\cot\theta)^2 - a^2}{b'^2(\cos\phi + \sin\phi'\cot\theta)^2 - b^2}.$$

One or both will be strained into circles provided the new angle between the conjugates is in one or both equal to a right angle. The inclination of the new position of a to x is given by ψ, $\tan\psi = z\sin\theta\sec\theta = z\tan\theta$; of a' similarly by $\tan\psi' = z\tan(\theta+\phi)$; and $1 + \tan\psi\tan\psi' = 0$ by hypothesis (if A becomes a circle); therefore $1 + z^2\tan\theta\tan(\theta+\phi) = 0$; this gives the additional condition that the ellipse with conjugates a, a' may be strained into a circle. Similarly for the other.

9538. (Professor WOLSTENHOLME, M.A., Sc.D.)—Two points, P, P′, are taken on the two parabolas $y^2 = 4ax$, $x^2 = 4ay$; prove that (1) if the tangents at P, P′ be parallel, the envelope of the straight line PP′ will be the curve whose equation is $x^3 + y^3 = 3axy$ (the Folium of Descartes); (2) if the tangents at P, P′ be at right angles, the envelope of PP′ will be a semi-cubical parabola, PP′ being always a normal to the parabola $(x-y)^2 = 16a(x+y+4a)$; (3) the locus of the intersection of the tangents at right angles to each other is the cissoid which is the pedal of the parabola $(x-y)^2 = 8a(x+y)$ with respect to its vertex; and (4) the area included between the Folium in (1) and either of the given parabolas is $\frac{1}{8}a^2$.

Solution by Rev. T. GALLIERS, M.A., *and* G. G. STORR, M.A.

1. P being $(a/m^2, 2a/m)$, P′ will be $(2am, am^2)$, and PP′, will have for its equation and envelope

$$(2m-m^4)\,x+(2m^3-1)\,y = 3am^2, \quad x^3+y^3-3axy = 0.$$

2. Here P′ will be $(-2a/m,\ a/m^2)$, and equation of PP′ and envelope

$$(\alpha)\ (2m^3-m^2)\,x-(2m^3+m^2)\,y+(4m^2+1)\,a = 0, \quad (x+y-4)^3 = 27a\,(x-y)^2,$$

a semi-cubical parabola. The equation of a normal to

$$(\beta)\ (x-y)^2 = 16a\,(x+y+4a) \ \text{ is } \ y = \mu x-4a\,(\mu-1)(\mu^2+1)/(\mu+1)^2,$$

and, since (α) may be expressed in this form, PP′ is normal to (β).

3. The equation of the specified locus may be shown to be

$$(\gamma)\ (x^2+y^2)(x+y)+a\,(x-y)^2 = 0 \ \text{ or } \ (X^2+Y^2)\,X+(2^{\frac{1}{2}}a)Y^2 = 0,$$

a cissoid.

Again, the equation of the specified pedal is the result of eliminating M between the equations

$$y = Mx+a\,(M+1)^2/(M-1) \ \text{ and } \ My+x = 0, \text{ identical with } (\gamma).$$

4. Let A = area included between Folium and the parabola $y^2 = 4ax$; then if $\quad r_1 = 3a\sin\theta\cos\theta/(\sin^3\theta+\cos^3\theta), \quad r_2 = 4a\cos\theta/\sin^2\theta,$

$$A = \tfrac{1}{2}\int_{\frac{1}{4}\pi}^{\frac{1}{4}\pi+\tan^{-1}(4)^{\frac{1}{3}}}(r_2^2-r_1^2)\,d\theta,$$

$$= 8a^2\int_{\frac{1}{4}\pi}^{\frac{1}{4}\pi+\tan^{-1}(4)^{\frac{1}{3}}}\cot^3\theta\,\mathrm{cosec}^2\theta\,d\theta-\frac{9a^2}{2}\int_{\frac{1}{4}\pi}^{\frac{1}{4}\pi+\tan^{-1}(4)^{\frac{1}{3}}}\frac{\tan^2\theta\sec^2\theta\,d\theta}{(1+\tan^3\theta)^2} = \tfrac{1}{8}a^2.$$

9674 & 9709. (Professor ABINASH BASÛ.) — If $\rho_1,\ \rho_2,\ \rho_3$ be the lengths of the three normals from $(x,\ y)$ to the parabola $y^2-4ax = 0$, prove that

$$\rho_1\rho_2\rho_3 = (y^2-4ax)\,\{y^2+(x-a)^2\}^{\frac{1}{2}}.$$

Solution by Professor MADHAVARAO; R. KNOWLES, B.A.; *and others.*

$$S = Y^2-4aX = 0, \quad S' = (X-x)^2+(Y-y)^2-\rho^2 = 0.$$

If $\Delta,\ \theta,\ \Theta',\ \Delta'$ be the invariants of these curves,

$$\Delta = -4a^2, \quad \Theta = -4a\,(a+x), \quad \Theta' = y^2-4ax-\rho^2, \quad \Delta' = -\rho^2.$$

If the circle $S' = 0$, having $x,\ y$ for the centre, touches the parabola, its radius ρ is the length of the normal from $x,\ y$.

The condition that S and S′ should touch each other is

$$\Theta^2\Theta'^2+18\Delta\Delta'\Theta\Theta'-27\Delta^2\Delta'^2-4\Delta\Theta'^3-4\Delta'\Theta^3 = 0,$$

which becomes

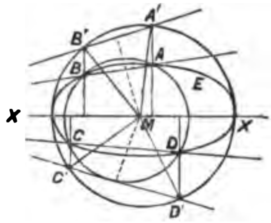
$$(a+x)^2(y^2-4ax-\rho^2)^2-18a\rho^2\,(a+x)(y^2-4ax-\rho^2)-27a^2\rho^4 \\ +(y^2-4ax-\rho^2)^3-16a\rho^2\,(a+x)^3 = 0,$$

from which

$$\rho_1^2\rho_2^2\rho_3^2 = (y^2-4ax)^2\,(a+x)^2+(y^2-4ax)^3 = (y^2-4ax)^2\,\{y^2+(a-x)^2\}.$$

9892. (Professor SYLVESTER, F.R.S.)—**Prove that, if any triangle of maximum area be inscribed in an ellipse, then the circle circumscribing it, the circles of curvature to the ellipse at its apices, and the ellipse itself will all five intersect each other in one and the same point.**

Solution by Professors SCHOUTE, WOLSTENHOLME, *and others.*



When A, B, C, D are four concyclic points of an ellipse E, the chords AB and CD are equally inclined on the arc XX. So, when E is considered as the orthogonal projection of a circle, the chords A′B′ and C′D′, of which AB and CD are the projections, admit the same property. This proves (see the equally inclined dotted lines that bisect the angles A′MB′ and C′MD′) that the sum of the eccentric angles of the four points A, B, C, D is a multiple of 2π.

Now, the triangles of maximum area are the projections of equilateral triangles in the circle. So the theorem is a consequence of the co-existence of the conditions

$$3\alpha+\beta\equiv 0,\ 3\left(\alpha+\tfrac{2}{3}\pi\right)+\beta\equiv 0,\ 3\left(\alpha+\tfrac{4}{3}\pi\right)+\beta\equiv 0,\ \alpha+\left(\alpha+\tfrac{2}{3}\pi\right)+\left(\alpha+\tfrac{4}{3}\pi\right)+\beta\equiv 0,$$

where the sign $\equiv$ means congruent with respect to 2π.

[See also the Solutions of Question 2223, on pp. 94, 95 of Vol. VII.]

9930. (CH. HERMITE, Membre de l'Institut.)—On donne les deux relations

$$\begin{vmatrix} a, & a', & x \\ b, & b', & y \\ c, & c', & z \end{vmatrix} = 0, \qquad \begin{vmatrix} a, & a', & x' \\ b, & b', & y' \\ c, & c', & z' \end{vmatrix} = 0;$$

on propose d'en déduire les suivantes

$$\begin{vmatrix} a, & x, & x' \\ b, & y, & y' \\ c, & z, & z' \end{vmatrix} = 0, \qquad \begin{vmatrix} a', & x, & x' \\ b', & y, & y' \\ c', & z, & z' \end{vmatrix} = 0.$$

Solution by THOMAS MUIR, LL.D.; R. H. W. WHAPHAM, B.A.; *and others.*

$$\begin{aligned} \text{Dat.}\quad & x\,|bc'| + y\,|a'c| + z\,|ab'| = 0, \\ \text{dat.}\quad & x'\,|bc'| + y'\,|a'c| + z'\,|ab'| = 0; \\ \text{ident.}\quad & a\,|bc'| + b\,|a'c| + c\,|ab'| = 0; \end{aligned}$$

therefore
$$\begin{vmatrix} x, & y, & \\ x', & y', & z' \\ a, & b, & c \end{vmatrix} = 0.$$

It is worth noting that the first two determinants and either of the

others are connected by a linear relation, viz.,

$$0 = \begin{vmatrix} a, & a' \\ b, & b' \end{vmatrix} \begin{vmatrix} a, & x, & x' \\ b, & y, & y' \\ c, & z, & z' \end{vmatrix} - \begin{vmatrix} a, & x \\ b, & y \end{vmatrix} \begin{vmatrix} a, & a', & x \\ b, & b', & y \\ c, & c', & z \end{vmatrix} + \begin{vmatrix} a, & x' \\ b, & y' \end{vmatrix} \begin{vmatrix} a, & a', & x \\ b, & b', & y \\ c, & c', & z \end{vmatrix},$$

and that the required result follows from this at once when $| a, b' | \neq 0$.

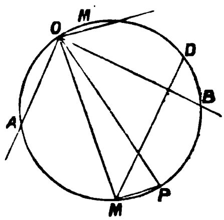
The above solution shows, however, that this condition is not necessary, for, if $| a, b' | = 0$, the three equations reduce to

$$\frac{x}{y} = \frac{x'}{y'} = \frac{a}{b} = \frac{| a, c' |}{| b, c' |},$$

a statement equivalent to saying that the elements of the first row of the third determinant are proportional to those of the second row, so that the result is the same as before.

9748. (Professor MANNHEIM.)—On donne un angle droit de sommet O. On décrit une circonférence passant par O, et l'on prend, sur cette courbe, un point M tel que les angles, compris entre les droites partant de ce point et aboutissant aux extrémités du diamètre qui contient O, aient pour bissectrices des parallèles aux côtés de l'angle donné. On demande le lieu de M, lorsqu'on fait varier la circonférence.

Solution by J. C. ST. CLAIR; G. G. STORR, M.A.; *and others.*



Let the sides of the fixed right angle cut any circle through O in A, B; let OP be the diameter through O, and let MD parallel to OA be the bisector of $\angle$ OMP. Since OMP is a right angle, $\angle$ OMD = 45°. And since MD is parallel to OA, $\angle$ OMD = AOM; therefore AOM = 45°, and OM bisects AOB. In like manner M′ lies on the external bisector of AOB. Hence the locus of M is the pair of lines bisecting the given angle AOB.

9937. (G. NIEWENGLOWSKI.)—Décomposer le produit $13 \times 37 \times 61$ en une somme de deux carrés, de quatre manières différentes.

Solution by R. W. D. CHRISTIE; Prof. NILKANTHA SARKAR; *and others.*

By Euler's theorem we have

$$(a^2+b^2)(c^2+d)^2 = (ac \pm bd)^2 + (ad \mp bc)^2 = A^2 + B^2,$$

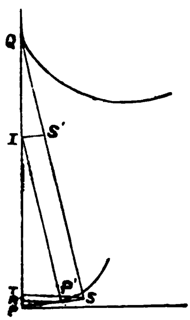
and $(a^2+b^2)(c^2+d^2)(e^2+f^2) = (A^2+B^2)(e^2+f^2) = (Ae \pm Bf)^2 + (Be \mp Af)^2$.

Thus $\quad 13 \times 37 \times 61 = (2^2+3^2)(6^2+1^2)(5^2+6^2) = \&c.$

$$= 171^2 + 10^2 = 21^2 + 170^2 = 165^2 + 46^2 = 75^2 + 154^2.$$

9901. (Professor WOLSTENHOLME, M.A., Sc.D.)—In a certain curve, the tangent line at a point Q is normal at P; prove that the *orthoptic locus* of the curve (locus of intersection of tangents at right angles) will touch the curve at P, and that its radius of curvature at P will be $QP^2/(QP + QI)$, where I is the centre of curvature of the curve at P. [Sign to be observed in the denominator.]

Solution by W. E. BRUNYATE; Prof. MATZ, M.A.; *and others.*



P is plainly a point on the locus. Let the consecutive tangent at Q meet the corresponding tangent at P′ in S. Then IP′ is parallel to QS. Draw P′R, ST perpendicular to PQ, and IS′ perpendicular to QS. Let $\angle PIP' = \phi$. Then in the limit S plainly moves up and lies on the tangent at P, or the locus touches the curve at P, and, if ρ be the radius of curvature,

$$\rho = \text{Lt}\,\frac{ST^2}{2PT} = \text{Lt}\,\frac{QP^2\phi^2}{2PT}.$$

But $PT = PR + RT$

$= PR + \text{projection of } IS' \text{ on } QP$

$= \tfrac{1}{2}PI\phi^2 + QI\phi^2,$

$$\therefore\quad \rho = \frac{PQ^2}{PI + 2QI} = PQ^2/(QP + QI);$$

attention being paid to the sign of QI.

8458. (W. J. GREENSTREET, B.A.)—A conic is inscribed in a triangle, and is such that the normals at the points of contact are concurrent. Find the locus of the point of concurrence, and show that the same cubic is the locus of the point of concurrence of normals drawn at the points of contact of the conic circumscribed about the triangle.

Solution by G. G. STORR, M.A.; Rev. T. GALLIERS, M.A.; *and others.*

Let $\quad L^2\alpha^2 + M^2\beta^2 + N^2\gamma^2 - 2MN\beta\gamma - 2NL\gamma\alpha - 2LM\alpha\beta = 0$ ....(1)

be the equation of the inscribed conic touching BC, CA, AB in D, E, F respectively. At D, $\alpha = 0$; hence, from (1), $M\beta - N\gamma = 0$.

Let $\quad l\alpha + m\beta + n\gamma = 0$ ....(2)

be the equation of the normal at D; then, since this line is perpendicular to $\alpha = 0$, $l - n\cos B - m\cos C = 0$, and since it passes through D, $mM + nN = 0$; hence

$$m : n = M : N,\quad l : n = (N\cos B - M\cos C)/N.$$

Substituting in (2), the equation of the normal at D becomes

$$M(\beta + \alpha\cos C) = N(\gamma + \alpha\cos B);$$

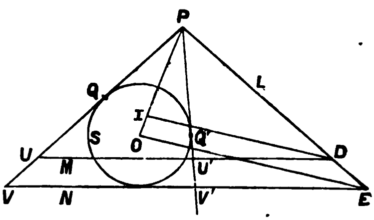
similarly $$N(\gamma+\beta\cos A) = L(\alpha+\beta\cos C)$$
and $$L(\alpha+\gamma\cos B) = N(\beta+\gamma\cos A)$$
are the equations of the normals at E and F. Eliminating L, M, and N between these three equations, the equation of the locus of concurrence becomes $$\alpha(\beta^2-\gamma^2)(\cos A-\cos B\cos C)+\beta(\gamma^2-\alpha^2)(\cos B-\cos C\cos A) +\gamma(\alpha^2-\beta^2)(\cos C-\cos A\cos B) = 0.$$

Again, let $\lambda\beta\gamma+\mu\gamma\alpha+\nu\alpha\beta = 0$ be the equation of the circumscribed conic, then the equation of the tangent at A is
$$\beta/\mu+\gamma/\nu = 0 \quad\quad (3),$$
and if $m\beta+n\gamma = 0$ be the equation of the straight line through A perpendicular to (3), we have
$$m\left\{1/\mu-(1/\nu)\cos A\right\} = -n\left\{(\cos A)/\mu-1/\nu\right\};$$
therefore (3) becomes $(1/\mu)(\beta\cos A+\gamma) = (1/\nu)(\gamma\cos A+\beta)$;
similarly $$(1/\nu)(\gamma\cos B+\alpha) = (1/\lambda)(\alpha\cos B+\gamma)$$
and $$(1/\lambda)(\alpha\cos C+\beta) = (1/\mu)(\beta\cos C+\alpha)$$
are the equations of the tangents at B and C. Eliminating λ, μ, ν between these three equations, we find the locus of concurrence to be the same as in the first case.

9912. (J. C. Malet, F.R.S.)—L and M are two right lines and S a circle, all situated in the same plane. If from a variable point on L two tangents be drawn to S, prove that the locus of the in-centre of the triangle formed by these tangents and the line M is a right line through the intersection of L and M.

Solution by R. F. Davis, M.A.; *and* Prof. Wolstenholme, M.A., Sc.D.

Let O be the centre of S. From any point P in L draw the tangents PQ, PQ′ meeting M in U and U′, and a tangent N (parallel to M) in V and V′ respectively. Also, let M, N meet L in D, E respectively. Then I, the in-centre of the triangle PUU′, lies on OP, and is such that



PI : PO = PU : PV = PD : PE.

Hence the locus of I is a straight line through D parallel to OE.

[Take the equations of S, L, M to be $x^2+y^2-c^2 = 0$, $px+qy+rc = 0$, $a-x = 0$; then, the in-centre is given by the equations
$$a-x = c-x\cos\theta-y\sin\theta = c-x\cos\phi-y\sin\phi,$$

θ, ϕ being connected by the equation

$$\begin{vmatrix} p, & q, & r \\ \cos\theta, & \sin^2\theta, & -1 \\ \cos\phi, & \sin^2\phi, & -1 \end{vmatrix} = 0; \quad \text{or } p\cos\tfrac{1}{2}(\theta+\phi)+q\sin\tfrac{1}{2}(\theta+\phi) + r\cos\tfrac{1}{2}(\theta-\phi) = 0.$$

At the in-centre, $x/\cos\frac{1}{2}(\theta+\phi) = y/\sin\frac{1}{2}(\theta+\phi) = c-a+x/\cos\frac{1}{2}(\theta-\phi)$, whence the locus of the in-centre is

$$px+qy+r(c-a+x) = 0, \quad \text{or} \quad px+qy+rc = r(a-x),$$

a straight line through (L, M). So the locus of that ex-centre which is opposite to the angular point on L is $px+qy+rc+r(a-x) = 0$, another straight line through (L, M), and these two loci form with L, M a harmonic pencil. The locus of the two remaining ex-centres will be much more complicated.]

9899. (Professor Hain.)—Soient A′, B′, C′ les symétriques d'un point quelconque P par rapport aux trois côtés d'un triangle ABC. (1) Lorsque P coïncide avec le centre d'un cercle tangent aux trois côtés de ABC, les droites AA′, BB′, CC′ concourent en un même point. (2) Lorsque le triangle ABC est équilatéral, les droites AA′, BB′, CC′ concourent en un même point, quel que soit le point P. *Corollaire.*—Dans tout triangle équilatéral, les symétriques, par rapport aux côtés, des droites joignant les sommets opposés à un même point, concourent également en un même point.

Solution by Professor Schoute.

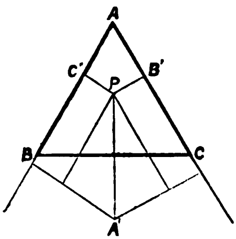


Fig. 1.

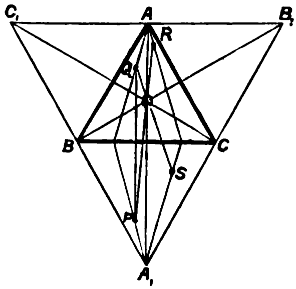


Fig. 2.

When A′ (Fig. 1) is the symmetrical point of P (x_1, y_1, z_1) with reference to BC, the normal coordinates of A′, as the figure shows, are

$$-x_1, \quad y_1+2x_1\cos C, \quad z_1+2x_1\cos B.$$

So the equation of the line AA′ is $y(z_1+2x_1\cos B) = z(y_1+2x_1\cos C)$.

Now, the three lines $\quad y(z_1+2x_1\cos B) = z(y_1+2x_1\cos C)$,

$$z(x_1+2y_1\cos C) = x(z_1+2y_1\cos A), \quad x(y_1+2z_1\cos A) = y(x_1+2z_1\cos B),$$

are concurrent under the condition

$$\frac{(z_1+2x_1\cos B)(x_1+2y_1\cos C)(y_1+2z_1\cos A)}{(y_1+2x_1\cos C)(z_1+2y_1\cos A)(x_1+2z_1\cos B)}=1.$$

The locus of the point P, for which the three lines AA′, BB′, CC′ are concurrent, therefore, is the cubic

$$\Sigma yz(y\cos C-z\cos B)+2\Sigma\cos B\cos C\,x(y^2-z^2)=0,$$

that contains the four in-centres. For an equilateral triangle ($\cos A = \cos B = \cos C = \frac{1}{2}$), the equation is an identity.

This last result and the corollary are geometrically evident. When P (Fig. 2) is given, and Q_a, Q_b, Q_c, and R are respectively its images with respect to BC, CA, AB, and its complementary point, then the lines AQ_a, BQ_b, CQ_c meet in the point inverse to R with reference to ABC. And when Q is given, and P_a, P_b, P_c, and S are its images in BC, CA, AB, and its anti-complementary point, then the lines A_1P_a, B_1P_b, C_1P_c meet in the point inverse to S with reference to $A_1B_1C_1$.

9920. (Frederick Purser, M.A.)—In a given quadrilateral is inscribed a fixed conic U, while a variable conic V is circumscribed to the same quadrilateral. Show that four of the chords of intersection of the fixed conic U with the varying conic V always touch a fixed conic S which is inscribed in the original quadrilateral.

Solution by W. S. McCay, M.A.; *and* Prof. Wolstenholme, Sc.D.

Let a variable circle pass through the foci F, F′ of a conic cutting it in two chords PP′, QQ′ parallel to tne axis, and cutting the transverse axis in S, S′; the chords PQ, PQ′, P′Q, P′Q′ touch a fixed confocal. It is at once seen that those chords are equally inclined to the focal vectors from P, P′, for S′ is the middle point of the arcs FF′. QQ′ on the circle. Let y_1, y_2 be the ordinates of P, Q, and C the centre of the conic. Then $y_1 \cdot CS = b^2$, $y_2 \cdot CS' = b^2$, since SP, SQ are tangent and normal, and PP′ the polar of S, therefore $y_1y_2 = b^4/c^2 =$ constant; but, for the four points F, P, F′, Q on a circle, the product of perpendiculars from PQ on FF′ is equal to the product of perpendiculars from FF′ on PQ, hence the axis minor of the confocal is fixed.

By projection of the circular points at infinity, we have the theorem.

[Let aa', bb', cc' be the diagonals of the quadrilateral, and let V pass through a, a', b, b'. Then, if we suppose a, a' projected into the isotropic points, U becomes a conic whose foci are b, b', c, c'; V a circle passing through the foci of U (b, b'). The common chords of U, V which are not parallel to bb' (*i.e.*, which do not pass through the intersection of bb', cc') touch a conic confocal with U. For, take

$$U \equiv x^2/a+y^2/b-1, \quad V \equiv x^2+y^2-2ky-c \text{ (where } c \equiv a-b\text{)},$$

and $px+qy=1$ a common chord; the complementary common chord will be $px-qy+1=0$, and

$$\lambda U+\mu V \equiv (px+qy-1)(px-qy-1);$$

so that $\lambda/a + \mu = p^2, \quad \lambda/b + \mu = -q^2, \quad \lambda + \mu c = 1,$
giving the tangential equation $p^2(c/b-1) + q^2(c/a-1) = 1/b - 1/a,$
and the envelope of the common chords is

$$x^2 \Big/ \left(\frac{c}{b} - 1\right) + y^2 \Big/ \left(\frac{c}{a} - 1\right) = 1 \Big/ \left(\frac{1}{b} - \frac{1}{a}\right),$$

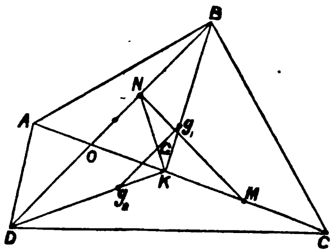
or

$$\frac{x^2}{a(c-b)} + \frac{y^2}{b(c-a)} = \frac{1}{a-b},$$

which is confocal with $x^2/a + y^2/b = 1$, that is to say, with U.]

9840. (Professor ABINASH BASU.)—ABCD is a quadrilateral, and O the point of intersection of AC and BD. From CO cut off CM equal to AO, and from BO cut off BN equal to DO. Prove that the centroid of the quadrilateral coincides with that of the triangle OMN.

Solution by ROSA H. WHAPHAM, B.A.; SARAH MARKS, B.Sc.; *and others.*



Let K be the mid-point of AC; g_1, g_2 the centroids of triangles ABC, ADC. Join g_1g_2; then g_1g_2 is parallel to BD. Join NK, meeting g_1g_2 in G; therefore G is centroid of OMN. Now

$$g_1G = \tfrac{1}{3}BN = \tfrac{1}{3}DO,$$
$$g_2G = \tfrac{1}{3}DN = \tfrac{1}{3}BO;$$
$$\therefore g_1G : g_2G = DO : BO$$
$$= \Delta ADC : \Delta ABC,$$

therefore G is the centroid of the quadrilateral, therefore centroids of the quadrilateral and the triangle OMN coincide.

9835. (Professor DE LONGCHAMPS.)—Résoudre l'équation

$$(ax+\beta)^3 + (a'x+\beta')^3 + x^3 = 3(ax+\beta)(a'x+\beta')x.$$

Déduire de là, en supposant $a = a' = 0$, une méthode élémentaire pour résoudre l'équation du troisième degré.

Solution by Professors DE WACHTER, BEYENS, *and others.*

Since $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$,
the equation $(ax+\beta)^3 + (a'x+\beta')^3 + x^3 - 3(ax+\beta)(a'x+\beta')x = 0$
reduces to

$$\left[(1+a+a')x + \beta + \beta'\right]$$
$$\times \left[(a^2+a'^2-aa'-a-a'+1)x^2 + (2a\beta + 2a'\beta' - a\beta' - a'\beta - \beta - \beta')x + \beta^2 + \beta'^2 - \beta\beta'\right] = 0.$$

Hence, the roots of this and of the proposed equation are (1) the value of x for which the factor $(1+\alpha+\alpha')\,x+\beta+\beta'=0$; (2) the two conjugate values of x which annihilate the second-degree factor. In particular, if $\alpha=\alpha'=0$, we have

$$x^3-3\beta\beta'x+\beta^3+\beta'^3=(x+\beta+\beta')\left[x^2-(\beta+\beta')\,x+\beta^2+\beta'^2-\beta\beta'\right]=0.$$

To identify this with the type-form $x^3+3px+2q=0$, we have to put $\beta\beta'=-p$, $\beta^3+\beta'^3=2q$. Hence we have

$$\beta=\left[q+(q^2+p^3)^{\frac{1}{2}}\right]^{\frac{1}{3}},\quad \beta'=\left[q-(q^2+p^3)^{\frac{1}{2}}\right]^{\frac{1}{3}}.$$

Thus, the required roots are, according to the above remark,

$$x_1=-(\beta+\beta'),\quad x_2=\beta\left[\tfrac{1}{2}(1+i\sqrt{3})\right]+\beta'\left[\tfrac{1}{2}(1-i\sqrt{3})\right],$$
$$x_3=\beta\left[\tfrac{1}{2}(1-i\sqrt{3})\right]+\beta'\left[\tfrac{1}{2}(1+i\sqrt{3})\right];$$

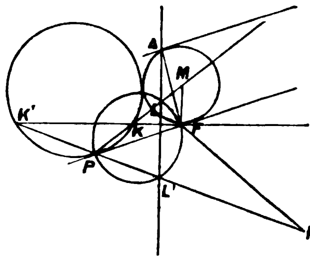
or, if we replace $\frac{1}{2}(1+i\sqrt{3})$ and $\frac{1}{2}(1-i\sqrt{3})$ respectively by ρ_1 and ρ_2,

$$x_1=(\beta+\beta')(-1),\quad x_2=\beta\rho_1+\beta'\rho_2,\quad x_3=\beta\rho_2+\beta'\rho_1;$$

where -1, ρ_1, and ρ_2 are the cubic roots of -1.

9692. (MAURICE D'OCAGNE.)—On donne deux points F et P, et une droite δ parallèle à FP. Si on considère une parabole variable, de foyer F, tangente à δ, les points de contact des tangentes menées de P à cette parabole sont sur un cercle fixe, passant par P.

Solution by A. PROVOST.



Let M and M′ be the points where the tangents from P touch the parabola; Δ, L, L′ the feet of the perpendiculars from F on the lines δ, PM, PM′. The tangent at the vertex of the parabola turns round Δ, and the axis cuts PL, PL′ at K, K′; also FK = FM and FK′ = FM′; and the similar triangles FPM and FPM′ give

$$\text{FM} : \text{FP} = \text{FP} : \text{FM}',$$
or
$$\text{FM}\,.\,\text{FM}' = (\text{FP})^2.$$

Besides, on the axis of a given parabola there are only two points, K and K′; consequently the locus of these points is the circle touching FP at P, cutting orthogonally the circle whose diameter is FP at the point where this circle itself is cut by circle whose diameter is ΔF.

But KL = ML, K′L′ = M′L′, we obtain, for the radius of the circle PKK′, $\frac{1}{2}(\text{PF})^2/\Delta\text{F}$; and easily

$$\rho=\text{PM}=2\text{PL}-\text{PK}=2\text{PF}\cos\alpha-\left[(\text{PF})^2/\Delta\text{F}\right]\sin\alpha,$$

where $\alpha=(\text{FPM})$, and which represents the required circle.

9740. (Professor HANUMANTA RAU, M.A.)—Four points are taken in one plane. Obtain a relation connecting the distances of the four points from one another.

Solution by Professor RAMASWAIN AIYAR.

Let A, B, C, D be the four points, and α, β, γ the angles which BC, CA, AB subtend at D; then either $\alpha+\beta+\gamma = 2\pi$, or the sum of two of these = the third; in any case

$$\sin\frac{\alpha+\beta+\gamma}{2}.\sin\frac{\alpha+\beta-\gamma}{2}.\sin\frac{\alpha-\beta+\gamma}{2}.\sin\frac{\beta+\gamma-\alpha}{2} = 0,$$

that is, $\quad 1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma = 0.$

Substitute for these cosines their values in terms of the distances of the points and the required relation is obtained.

9606. (BELLE EASTON.)—Solve (1) the equations

$$4(x-a)^2 = 9(x-b)(a-b) \qquad\qquad (\alpha);$$

$$x(y+z-x) = a^2,\; y(z+x-y) = b^2,\; z(x+y-z) = c^2 \qquad\qquad (\beta);$$

$$u(2a-x) = x(2a-y) = y(2a-z) = z(2a-u) = b^2 \qquad\qquad (\gamma);$$

and (2) prove that, in (γ), unless $b^2 = 2a^2$, $x = y = z = u$, but that, if $b^2 = 2a^2$, the equations are not independent.

Solution by Rev. J. L. KITCHIN, M.A.; BELLE EASTON; *and others.*

(α) $\qquad 4\{(x-b)-(a-b)\}^2-9(x-b)(a-b) = 0,$

or $\qquad 4(x-b)^2-17(x-b)(a-b)+4(a-b)^2 = 0,$

therefore $\qquad \{4(x-b)-(a-b)\}\{(x-b)-4(a-b)\} = 0,$

whence $\qquad x = \frac{1}{4}(a+3b)$ or $x = 4a-3b.$

(β) From (1) and (2), $(x-y)\{x+y-z\} = b^2-a^2$, therefore, by (3), $(x-y)/z = (b^2-a^2)/c^2$. Putting this value for z in (2) and (1),

$$\frac{x}{a^2(b^2+c^2-a^2)} = \frac{y}{b^2(a^2+c^2-b^2)} = \frac{z}{c^2(a^2+b^2-c^2)},$$

by symmetry; and the result follows immediately.

(γ) Eliminating μ, y, z, we get $(2a^2-b^2)(x^2-2ax+b^2) = 0.$

If b^2-2a^2 be real, $\quad x^2-2ax+b^2 = 0,\;$ and $\; x = a\pm(a^2-b^2)^{\frac{1}{2}}.$

Now $\qquad 2a-x = a\mp(a^2-b^2)^{\frac{1}{2}},$

therefore $\qquad \mu = \dfrac{b^2}{a\mp(a^2-b^2)^{\frac{1}{2}}} = a\pm(a^2-b^2)^{\frac{1}{2}};$

therefore $= y = z$. (2) follows obviously.

9573. (Professor HUDSON, M.A.)—A particle P describes a rectangular hyperbola under a force from the centre C; a point CY is taken in CP so that CY . CP = CA^2; prove that, if v be the rate at which P and Y separate,
$$v^2 = \mu\, CP^2 \left(1-\frac{CA^2}{CP^2}\right)\left(1+\frac{CA^2}{CP^2}\right)^3.$$

Solution by Rev. T. GALLIERS, M.A.; *the* PROPOSER; *and others.*

Let the rectangular hyperbola be $r^2\cos 2\theta = a^2$; then we have
$$CY = a^2/r, \text{ therefore } PY = r - a^2r^{-1} \text{ ....(1, 2).}$$

Now
$$v = \frac{d\,(PY)}{dt} \quad \frac{d\,(PY)}{dr}\cdot\frac{dr}{dt} = \frac{d\,(PY)}{dr}\cdot\frac{dr}{d\theta}\cdot\frac{d\theta}{dt};$$

and
$$\frac{d\,(PY)}{dr} = 1 + \frac{a^2}{r^2}.$$

From Dynamics $\frac{d\theta}{dt} = h/r^2$; and $r\frac{dr}{d\theta} = a^2\sec 2\theta\tan 2\theta$; hence
$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^6}{a^4}\left(1-\frac{a^4}{r^4}\right),$$
$$v^2 = \frac{h^2}{a^4}r^2\left(1+\frac{a^2}{r^2}\right)^3\left(1-\frac{a^2}{r^2}\right) \text{ ....(3).}$$

From the equation for central forces, $h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right) = P$, $P = -\frac{h^2}{a^4}r$. Putting $h^2 = \mu a^4$ in (3), we obtain the result required.

[If the force from the centre be μCP, then the velocity at P is $\sqrt{\mu}$CP. Draw CU perpendicular to the tangent at P, then CU = Cy. In the short time T let P come to P′, y to y', and draw y_n, P_m perpendicular to Cy'P′; then $yy'n$, PP′m are ultimately similar to CPU;

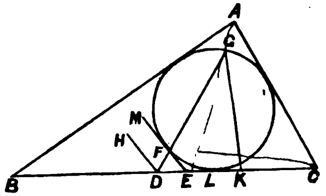
therefore $\frac{y'n}{yy'} = \frac{P'm}{PP'} = \frac{PU}{CP}$ ultimately; also $\frac{yy'}{PP'} = \frac{Cy}{CP}$ ultimately.

Now $v = lt\frac{P'y'-Py}{T} = lt\frac{PP'}{T} + \frac{P'm+Y'n}{PP'} = \sqrt{\mu}CP\frac{PU}{CP}\left(1+\frac{Cy}{CP}\right),$

$$\therefore\ v^2 = \mu CP^2\left(1-\frac{CA^4}{CP^4}\right)\left(1+\frac{CA^2}{CP^2}\right)^2 = \mu CP^2\left(1-\frac{CA^2}{CP^2}\right)\left(1+\frac{CA^2}{CP^2}\right)^3.]$$

9728. (J. YOUNG, M.A.)—Prove that the nine-point circle and the in-circle touch each other, and that the point of contact is in the production of the line joining the mid-point of the base with the point of contact of the tangent to the in-circle drawn from the point where the internal bisector of the vertical angle cuts the base.

Solution by the PROPOSER; E. M. LANGLEY, M.A.; *and others.*



The line EFM (the tangent of the question) and DH (the tangent to the nine-point circle at mid-point of base) are parallel, the inclination of each to the base being the difference of the base angles. Let DF produced meet the inscribed circle in G; join G with K, the foot of the perpendicular from A on BC. It can be shown, by joining D with the foot of the perpendicular from C on AE, that the rectangle DK.DE is the square on half the difference of the sides AB, AC, that is of DL; L being the point of contact of BC with the inscribed circle. Hence the four points K, E, F, G are concyclic; therefore the angles HDB, KGD are equal, which proves that G lies on the nine-point circle. Also a line through G making with FG an angle equal to GFM or GDH is a tangent to both circles, proving that they touch at G.

[A similar proof applies to the ex-circle, L being the same distance to the left of D, and F below the base.]

9907. (Professor MADHAVARAO.)—Prove that the locus of the centre of a circle of invariable radius r, intersecting the ellipse $x^2/a^2+y^2/b^2=1$, so that a common chord always passes through a fixed point $(\alpha\beta)$, is

$$\left\{(x-\alpha)^2+(y-\beta)^2-r^2\right\}^3$$
$$+(x^2+y^2-a^2-b^2-r^2)\left\{(x-\alpha)^2+(y-\beta)^2-r^2\right\}^2(\alpha^2/a^2+\beta^2/b^2-1)$$
$$-\left\{(x^2-r^2)\,b^2+(y^2-r^2)\,a^2-a^2b^2\right\}\left\{(x-\alpha)^2+(y-\beta)^2-r^2\right\}(\alpha^2/a^2+\beta^2/b^2-1)^2$$
$$-a^2b^2r^2(\alpha^2/a^2+\beta^2/b^2-1)^3=0.$$

Solution by ROSA H. WHAPHAM; BELLE EASTON; *and others.*

Let $\quad S\equiv(x-h)^2+(y-k)^2-r^2=0,\quad S\equiv x^2/a^2+y^2/b^2-1=0;$

then the equation of the three pairs of lines joining the points of intersection of $S'=0$ and $S=0$, is

$$\Delta S'^3-\Theta S'^2S+\Theta'S'S^2-\Delta'S^3=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where Δ, Θ, Θ', Δ' are the invariants of the system. Hence,

$$\Delta=-1/a^2b^2,\quad \Theta=(h^2+k^2-a^2-b^2-r^2)/a^2b^2;$$
$$\Theta'=\left\{(h^2-r^2)\,b^2+(k^2-r^2)\,a^2-a^2b^2\right\}/a^2b^2,\quad \Delta'=-r^2.$$

Substituting in (1),

$$S'^3+(h^2+k^2-a^2-b^2-r^2)\,S'^2S-\left\{(h^2-r^2)\,b^2+(k^2-r^2)\,a^2-a^2b^2\right\}S'S^2$$
$$-a^2b^2r^2S^3=0.$$

Since $\alpha\beta$ is on one of these chords, we get for the locus of (h, k) the result stated in the question.

9913. (A. Russell, B.A.)—If a polygon be inscribed in a circle, prove that $\Sigma (a_{r-1}^2 - a_r^2) \cot A_r = 0$, where a_{r-1}, a_r are two consecutive sides, and A_r the included angle.

Solution by Professor Wolstenholme, M.A., Sc.D.

Let the radius of the circle be unity, $2\theta_1$, $2\theta_2$, $2\theta_3$ ... the angles subtended at the centre by the sides; then

$$a_r = 2\sin\theta_r, \quad a_{r-1}^2 - a_r^2 = 4\sin(\theta_{r-1} - \theta_r)\sin(\theta_{r-1} + \theta_r),$$

and $$A_r + \theta_{r-1} + \theta_r = \pi, \quad \cot A_r = -\cot(\theta_{r-1} + \theta_r),$$

$$(a_{r-1}^2 - a_r^2)\cot A_r = -4\sin(\theta_{r-1} - \theta_r)\cos(\theta_{r-1} + \theta_r) = -2(\sin 2\theta_{r-1} - \sin \theta_{2r}),$$

so that $$\Sigma (a_{r-1}^2 - a_r^2)\cot A_r \equiv 0.$$

9919. (R. A. Roberts, M.A.)—Show that the locus of the intersection of rectangular tangents of the cubic $xy^2 = 4p^3$ is the circle $x^2 + y^2 = 3px$, the axes of coordinates being rectangular.

Solution by W. E. Brunyate; Sarah Marks, B.Sc.; *and others.*

Let any point be $x = p/a^2$, $y = 2pa$, and then the tangent at a is $xa^2 + y/a = 3p$, and for a perpendicular tangent a', $a^3 a'^3 - 1$; hence the locus is given by

$$xa^2 + y/a = 3p = x/a^2 - ya, \quad i.e., \quad (x^2 + y^2)(1 + a^6) = 9p^2(a^2 + a^4),$$

and $$x^6(1 + a^6) = 3p(a^2 + a^4), \quad \text{or} \quad x^2 + y^2 = 3px.$$

9905. (Professor Genese, M.A.)—Prove the following extension of a theorem due to Monsieur le Docteur Laisant (*Mathesis*, Question 628):—The locus of a point, the product of whose distances from the vertices of a regular polygon is constant, is given by the equation

$$\rho^{2n} - 2\rho^n a^n \cos n\theta = b^{2n}.$$

Solution by Professors Wolstenholme, Beyens, *and others.*

Suppose A_1, A_2, ... A_n to be the vertices of a regular polygon inscribed in a circle of radius a, whose centre is O; let $OP = r$, $\angle POA_1 = \theta$, $\alpha = 2\pi/n$. Then $A_1P^2 = r^2 + a^2 - 2ar\cos\theta$,

$$A_2P^2 = r^2 + a^2 - 2ar\cos(\theta + \alpha), \quad A_3P^2 = r^2 + a^2 - 2ar\cos(\theta + 2\alpha), \text{ \&c.},$$

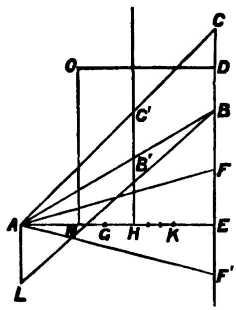
so that, if the product of the distances A_1P, A_2P, &c. be c^n, the equation of the locus of P will be

$$(r^2 + a^2 - 2ar\cos\theta)\left\{r^2 + a^2 - 2ar\cos(\theta + \alpha)\right\}\left\{r^2 + a^2 - 2ar\cos(\theta + 2\alpha)\right\} \ldots$$
$$\text{to } n \text{ factors} = c^{2n},$$

or $$r^{2n}+a^{2n}-2a^n r^n \cos n\theta = c^{2n},$$
by the well-known identity. This may be written
$$r^{2n}-2a^n r^n \cos n\theta = b^{2n},$$
but b^{2n} may be either positive or negative.

9853. (Professor DÉPREZ.)—On considère tous les triangles qui ont un sommet fixe A, et les deux autres sommets B, C sur une droite donnée; la longueur BC est également donnée. Trouver les lieux géométriques du centre du cercle circonscrit et du centre du cercle des neuf points, ainsi que les enveloppes des hauteurs issues de B et C.

Solution by W. S. FOSTER; Professor CHAKRAVARTI; *and others.*



Let AE be perpendicular to BC; O the centre of the circle about ABC; OD, OM at right angles to BC, AE; G the point where the circumscribing circle cuts AE; let
$$EF = \tfrac{1}{2}BC = EF'.$$
Then OD, OM bisect BC and AG. Then
$$EB \,.\, EC = EG \,.\, EA;$$
$$OM^2 = EF^2 + EA^2 - 2EA \,.\, AM = 2AE \,.\, MK,$$
if K be the centre of AFF'. Hence the locus of M is a parabola.

The nine-point circle will pass through E, B'C', and $B'C' = \tfrac{1}{2}BC$, and the centre, as in the former case, will lie on a parabola.

Take AL, perpendicular to AE, and equal to BC. The perpendicular from B on AC will be perpendicular to BL, and the envelope of this perpendicular will be a parabola, focus L, and EBC, the tangent at the vertex.

9833. (Professor WETZIG.)—Soient AA', BB', CC' les hauteurs du triangle ABC, et K, K', K'', K''' les points de LEMOINE des triangles ABC, AB'C', A'BC', ABC'. Démontrer que K est au milieu des perpendiculaires abaissées de K' sur BC, de K'' sur CA, de K''' sur AB.

Solution by R. TUCKER, M.A.; R. KNOWLES, B.A.; *and others.*

Let the tangents to the circumcircle at B, C, intersect in T, and let AT cut BC in S, and B'C' in M'; then K is on AS, and AM' is the median of AB'C'. Again, let AM, the median of ABC, cut BC in M and B'C' in S'; then AS' is the symmedian of AB'C' through A, and therefore

contains K′. Let TM cut the circumcircle in X, Y; then, since T, X, M, Y is a harmonic range, we have

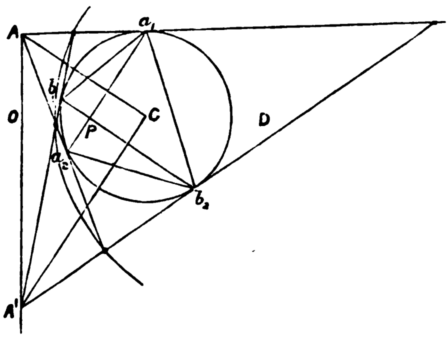
$$TX : XM = TY : MY = TC : CM = AB : AB' = AK : AK' = AS : AS'.$$

Therefore KK′ and SS′ are parallel to XY. But, by a well-known property of the symmedian point, MK bisects the perpendicular from A on BC; hence K is the mid-point of perpendicular from K′ on BC, and so for the other points.

9772. (G. Rumilly.)—On donne deux droites rectangulaires OX, OY, et un cercle dont le centre C est sur OX. Autour du point C tourne un angle droit dont les côtés coupent OY en A et A′. De ces points on mène des tangentes au cercle: lieu des points d'intersection de ces tangentes.

Solution by J. C. St. Clair; Fannie H. Jackson; *and others.*

Let the chord of contact of A be a_1, a_2, and that of A′, b_1, b_2. Since AC, A′C are at right angles, so also are the chords of contact, which



always pass through P, the pole of AA′. Hence the sides of the inscribed quadrilateral $a_1 b_1 a_2 b_2$ always subtend a right angle at a fixed point P, and therefore (Casey's *Sequel to Euclid*, Bk. vi., Sec. iv., Prop. 13) the locus of their poles, *i.e.*, the intersections of the tangents from A, A′, is a circle.

3822. (Artemas Martin, M.A.)—Find the mean distance of the centre of the base of a given right cone (1) from all points in its curved surface, and (2) from all points within the cone.

Solution by D. Biddle.

1. Let unity represent the distance from the apex of the cone to the circumference of its base, a = the height of the cone, r = the radius of its base, and x = the distance, from the apex, of a point on the curved surface. Then the distance of this point from the centre of the base $= (a^2-2a^2x+x^2)^{\frac{1}{2}}$, and we obtain the answer as follows:—

$$A_1 = \int_0^1 x\,(a^2-2a^2x+x^2)\,dx \div \int_0^1 x\,dx = 2\int_0^1 x\,(a^2-2a^2x+x^2)\,dx$$

$$= 2\left\{\frac{(1-a^2)^{\frac{3}{2}}-a^3}{3} + a^2\int_0^1 (a^2-2a^2x+x^2)\,dx\right\}$$

$$= 2\left\{\frac{(1-a^2)^{\frac{3}{2}}-a^3}{3} + a^2\left(\frac{2\,(1-a^2)^{\frac{3}{2}}+2a^3}{4} + \frac{a^2-a^4}{2}\int_0^1 \frac{dx}{(a^2-2a^2x+x^2)^{\frac{1}{2}}}\right)\right\}$$

$$= (1-a^2)^{\frac{3}{2}}\left(\tfrac{2}{3}+a^2\right) - a^3\left(\tfrac{2}{3}-a^2\right) + (a^4-a^6)\log\frac{1-a^2+(1-a^2)^{\frac{1}{2}}}{a-a^2}$$

$$= r^3\left(\tfrac{2}{3}+a^2\right) - a^3\left(\tfrac{2}{3}-a^2\right) + a^4r^2\log\frac{r+r^2}{a-a^2}.$$

2. In order to find A_2, suppose the solid cone to consist of an infinite number of similar hollow cones, one within the other. Then the mean distance from the common centre of the base, of points on the surface of each such cone, will be proportional to any one of its linear dimensions, whilst its curved surface will be as the square. Consequently,

$$A_2 = \int_0^{A_1} x^2\,x\,dx \div \int_0^{A_1} x^2\,dx = \tfrac{3}{4}A_1.$$

Examples.—Where $a = r = \sqrt{\frac{1}{2}}$, we have

$$A_1 = \tfrac{1}{2}\sqrt{\tfrac{1}{2}} + \tfrac{1}{8}\log\frac{\sqrt{\frac{1}{2}}+\frac{1}{2}}{\sqrt{\frac{1}{2}}-\frac{1}{2}} = \cdot573897, \quad A_2 = \tfrac{3}{4}A_1 = \cdot430423;$$

where $a = \cdot8$, $r = \cdot6$, we have

$$A_1 = \cdot216\,(1\cdot30\dot{6}) - \cdot512\,(\cdot02\dot{6}) + \cdot147456\log 6 = \cdot532793,$$

$$A_2 = \tfrac{3}{4}A_1 = \cdot399595.$$

9702. (Professor Hudson, M.A.)—A heavy particle is projected upwards from the vertex, within a smooth parabola whose axis is horizontal, with a velocity due to a fall down the latus rectum ($4a$). Investigate the subsequent motion, and show that the particle impinges upon the parabola again, at a distance $3a\sqrt{13}$ from the vertex, with a velocity that bears to the velocity of projection the ratio $\sqrt{5} : \sqrt{2}$.

Solution by Professor Madhavarao; Rev. T. Galliers; *and others.*

Let S be focus, P the point where the particle leaves the curve, PN the ordinate, v = initial velocity, v' = velocity at P, $2\theta = \angle$ PSN; then, for the equation of motion when the constraint ceases, ρ being radius of curvature at P, $v'^2/\rho = g\sin\theta$, and $v'^2 = v^2 - 2g\,\text{PN} = 8ag - 4ag\tan\theta$;

therefore $$4ag\,(2-\tan\theta) = g\rho\sin\theta = 2ag\,\frac{\sin\theta}{\cos^3\theta},$$

therefore $$\tan^3\theta + 3\tan\theta - 4 = 0, \quad \text{whence } \theta = \tfrac{1}{4}\pi.$$

If x and $-y$ be the ordinate and abscissa of the particle at the end of time t, we have $$x = a + \frac{v't}{\sqrt{2}}, \quad y = -\frac{v't}{\sqrt{2}} + \tfrac{1}{2}gt^2 - 2a.$$

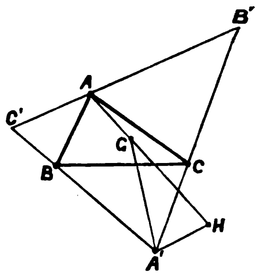
If the particle meet the curve again, we have $y^2 = 4ax$, from which $t = 4\sqrt{\dfrac{2a}{g}}$, $y = 6a$, $x = 9a$; hence the distance from vertex $= 3a\sqrt{13}$.

If V be the velocity of impinging,

$$V^2 = \frac{v'^2}{2} + \left(\frac{v'}{\sqrt{2}} - gt\right)^2 = 20ag, \quad \text{therefore } \frac{V}{v} = \frac{\sqrt{5}}{\sqrt{2}}.$$

9900. (Professor Neuberg.)—A un triangle donné ABC, on circonscrit tous les triangles A′B′C′ qui ont pour centre de gravité un point donné G. Trouver les lieux décrits par les sommets A′, B′, C′.

Solution by R. F. Davis, M.A.



Let the sides B′C′, C′A′, A′B′ pass through A, B, C respectively. Join AG, and produce it to H, so that

$$GH = 2AG.$$

Then, since G is the centroid of the triangle A′B′C′, and A′H is parallel to the base B′C′,

$$A' \{BGCH\} \text{ is harmonic;}$$

and the locus of A′ is a conic through B, G, C, H.

9660. (Professor Hanumanta Rau, M.A.)—Show that the sum to n terms of the series $2 + 0 + 7 - 4 + 21 - 26 + 71 - \dots$ is

$$S_n = \tfrac{1}{2}\left[n(n+1)\right] + \tfrac{1}{3}\left[(-2)^n - 1\right].$$

Solution by W. J. Greenstreet, M.A.

In usual way n^{th} term is found to be of form $A + Bn + C(-2)^{n-1}$, and $A = 0$, $B = 1$, $C = 1$, therefore n^{th} term is $n + (-2)^{n-1}$, therefore, &c.

9892. (Professor SYLVESTER, F.R.S.)—Prove that, if any triangle of maximum area be inscribed in an ellipse, then the circle circumscribing it, the circles of curvature to the ellipse at its apices, and the ellipse itself will all five intersect each other in one and the same point.

Solution by R. F. DAVIS, M.A.; Rev. T. GALLIERS, M.A.; *and others.*

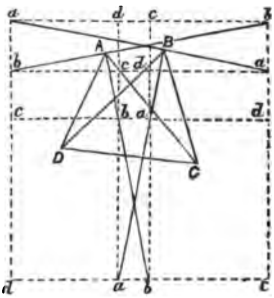
If P be a point on a circle whose eccentric angle (measured from a fixed radius OA in a fixed sense of rotation) is α, then the eccentric angle of a point Q, such that PQ and the tangent to the circle at P are equally inclined to OA, is easily found to be $= -3\alpha$.

For the three vertices P_1, P_2, P_3 of a maximum inscribed (equilateral) triangle whose eccentric angles are α, $\alpha + \frac{2}{3}\pi$, $\alpha + \frac{4}{3}\pi$, the associated points Q_1, Q_2, Q_3 each coincide with the point Q whose eccentric angle $= -3\alpha\pi$ and the sum of the eccentric angles of the four points P_1, P_2, P_3, Q $= 2$.

Hence, projecting orthogonally, &c.

9948. (W. S. M'CAY, M.A.)—A, B, C, D are four points on a circle. Omitting each point in turn, we have four triangles; prove that the sixteen centres of the circles touching the sides of these triangles lie in fours on four parallel lines and also in fours on four perpendicular lines, and that the two sets of lines are parallel to the bisectors of the angles between AC and BD.

Solution by FREDERIC R. J. HERVEY; W. J. JOHNSTON, M.A.; *and others.*



Let the four centres of the triangle formed by leaving out A be denoted by a, and so on. If B move to A, BC, BD turn through equal angles, and so therefore do the bisectors of the angles BCD, BDC. Hence the bisectors of the angles at the common base of the triangles ACD, BCD determine two concyclic and rectangular sets of points $abab$; aa, bb, being diameters. The diagonals aa, bb are in one rectangle, the internal bisectors, in the other the external bisectors, of the angles A, B; the bisectors of the angles between them (and hence the sides ab) are evidently parallel to those of the angles between AC, BD and AD, BC.

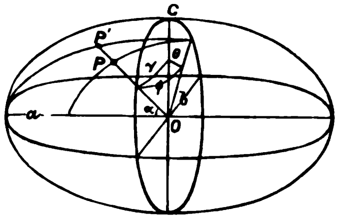
If A, B were on opposite sides of CD, we should have one internal and one external bisector, with the same result.

By considering in the same way the pairs of triangles having the common bases DB and BC, we shall find that each of the six lines aa is a diagonal of one such rectangle. Hence, if parallels to the bisectors of the angles between AC, BD, &c. be drawn through each of the points a, their remaining intersections will be the centres b, c, d.

9758. (J. O'Byrne Croke, M.A.)—If every section of an ellipsoid through the axis of x be contracted into a circle with the centre of the ellipsoid as centre, and the semi-conjugate axis of the section as radius, prove that the ellipsoid becomes thus contracted into the quadric surface

$$(x^2+y^2+z^2)\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) = y^2+z^2.$$

Solution by the Proposer; Prof. Sarkar; *and others.*



Let ρ be the radius vector to any point P′ on the ellipsoid; and $r = \text{OP}$, where P is the corresponding point in the locus.

Then, if α, β, γ be the direction angles of OP, θ the angle between the axis of z and the trace of the central section through P′ on the plane of yz, and ϕ the angle between this trace and OP′, we have $\cos\alpha = \sin\phi$; $\cos\beta = \sin\theta\cos\phi$; $\cos\gamma = \cos\theta\cos\phi$; $x = r\cos\alpha$; $y = r\cos\beta$; $z = r\cos\gamma$.

And
$$\rho^2\left(\frac{\cos^2\alpha}{a^2}+\frac{\cos^2\beta}{b^2}+\frac{\cos^2\gamma}{c^2}\right) = 1;$$

$$\rho^2\left(\frac{\sin^2\phi}{a^2}+\frac{\cos^2\phi}{r^2}\right) = 1; \quad r^2\left(\frac{\sin^2\theta}{b^2}+\frac{\cos^2\theta}{c^2}\right) = 1.$$

$$\therefore\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = r^2\left(\frac{\sin^2\phi}{a^2}+\frac{\cos^2\phi}{r^2}\right); \quad \frac{r^2}{\cos^2\phi}\left(\frac{\cos^2\beta}{b^2}+\frac{\cos^2\gamma}{c^2}\right) = 1 \ldots (1, 2).$$

$$\therefore\quad \frac{1}{\cos^2\phi}\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) = 1; \quad \cos^2\phi = \frac{y^2}{b^2}+\frac{z^2}{c^2}; \quad \sin^2\phi = 1-\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right);$$

$$r^2 = x^2+y^2+z^2.$$

And by substituting these values of r^2, $\cos^2\phi$, and $\sin^2\phi$ in (1), and simplifying, we have

$$(x^2+y^2+z^2)\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)+x^2 = 0,$$

the equation of the quadric surface which is the locus of P.

9960. (E. Lemoine.)—On circonscrit à toutes les ellipses homofocales de foyers F et F′ des *rectangles* dont les directions des côtés sont données; démontrer que tous les points de contact, quelle que soit l'ellipse à laquelle est circonscrit un rectangle, appartiennent à une même hyperbole équilatère qui passe par F et par F′, qui a pour asymptotes les parallèles aux côtés des rectangles menées par le centre des ellipses. Le lieu des sommets de ces hyperboles équilatères, quand la direction des côtés des rectangles varie, est une lemniscate de Bernoulli.

Solution by Professor SCHOUTE; R. F. DAVIS, M.A.; *and others.*

La droite a (Fig. 1) de direction donnée est touchée par une des coniques de foyers F et F′ au point P, pour lequel PF et PF′ font de part et d'autre des angles égaux avec a. Le lieu de P est donc en même temps

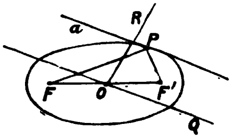


Fig. 1.

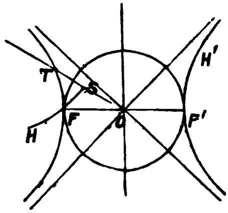


Fig. 2.

le lieu du point d'intersection de deux droites antiparallèles par rapport à la direction donnée, dont l'une passe par F et l'autre par F′, c.-à-d. l'hyperbole équilatère à centre O, qui passe par F et F′ et dont les asymptotes OQ et OR sont parallèles aux côtés des rectangles déterminés par la direction donnée (problème du réverbère hissé).

Les hyperboles équilatères passant par F et F′ et à centre O forment un faisceau dont les quatre foyers communs des coniques sont les points de base. Soit S un des sommets d'une de ces hyperboles équilatères, H et T un des points d'intersection de OS et de l'hyperbole équilatère H′, dont FF′ est l'axe réel. Comme on a évidemment OS : OF = OF : OT, le lieu de S est la transformée par rayons vecteurs réciproques à centre O et puissance OF^2, c.-à-d. une lemniscate de Bernoulli, etc.

[Let ORPY be a side of the rectangle touching at P one of the system of confocal ellipses. Draw the perpendiculars FY, CR upon the tangent, and CE upon FY; so that CR, CE are the first directions. Let the tangent and normal at P meet the axis in T, G respectively. Then CT : CF = CF : CG, or YE : FE = YR : PR, and CR . PR = FE . CE = const., so that P lies on a rectangular hyperbola, having CR, CE for asymptotes and passing through F. When the directions vary, if σ, σ' be the foci of any one of the rectangular hyperbolas, $\sigma F . \sigma' F = CF^2 =$ const., or $\sigma F . \sigma F'$ is const. Hence the locus of σ (and wherefore also of its corresponding vertex) is a lemniscate.]

9951. (D. BIDDLE.)—Required that function of x which, when x in replaced by 1, 2, 3, 4, yields respectively 0, $\frac{2}{3}$, $\frac{17}{36}$, $\frac{641}{900}$.

Solution by R. KNOWLES, B.A.; R. W. D. CHRISTIE; *and others.*

Let $x^5 + bx^4 + cx^3 + dx^2 + ex$ be the function of x required; then, by putting $x = 1, 2, 3, 4$ respectively, four equations are formed from which b, c, d, e are determined. Substituting these values, the function becomes

$$(64800x^5 - 647577x^4 + 2263362x^3 - 3221847x^2 + 1541262x) / 64800.$$

[Applying the successive differences 2, 13, 596, and 1, 2, 91 to the series

$$u+(x-1)\,\Delta u+\frac{(x-1)(x-2)}{2\,!}\,\Delta^2 u+\&c.,$$

Mr. Christie finds that the conditions of the question are satisfied by

$$(596x^3-3537x^2+6451x-3510)/9\,(91x^3-540x^2+989x-540).$$

Mr. Dickson finds the function

$$(-34x^3+279x^2+601x-846)/5400.$$

The Proposer adds that the successive amounts in the question are given by $\Sigma\, 2\,(n-1)/(n+1)^2$.]

9035. (Professor Cavallin, M.A.)—If A and B are two luminous points whose intensities are as $n:1$, and P a point in an ellipse of which they are the foci, show (1) that the illumination of the curve at P is a maximum or a minimum, when

$$AP\,[5\,(AP)/(BP)-1] = n\,.\,BP\,[5\,(BP)/(AP)-1]\,;$$

(2) determine which it is, and show that for such points AP must be $>\frac{1}{6}$ and $<\frac{5}{6}$ major axis; and (3) show that, by increasing the value of n, the above value of AP increases.

Solution by Rev. T. Galliers, M.A.; G. G. Storr, M.A.; *and others.*

Let AP $= r$, BP $= r'$, I $=$ illumination of the curve at P, then, by Optics, it may be shown that $\;\; I = \lambda\,(nr'^2+r^2)/(rr')^{\frac{3}{2}}$,

where $\lambda =$ constant; also $r+r' = 2a$.

Hence $\qquad dI/dr = \lambda\left\{(5r-r')\,r^2-(5r'-r)\,nr'^2\right\}/(rr')^{\frac{5}{2}}$

therefore I is a maximum or minimum when

$$r\,(5r/r'-1) = nr'\,(5r'/r-1) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots \;(1),$$

Putting $r'/r = \rho$ in (1), we get $\;\; n\rho^2\,(5\rho-1)+\rho-5 = 0 \ldots\ldots\ldots\ldots\;(2)$;

therefore r must lie between $\frac{1}{6}\,.\,2a$ and $\frac{5}{6}\,.\,2a$.

To find whether I is a maximum or minimum for points determined by (1). We shall find that the sign of d^2I/dr^2 depends on the sign of

$$40\rho'^2+32\rho'+3n-5,$$

where $\rho' = \rho^{-1}$, and the value of ρ satisfies (2).

Hence we may infer that d^2I/dr^2 is positive for values of r determined by (1). Hence I is a *minimum* for such points.

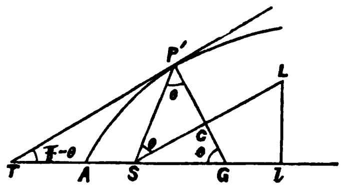
(3) Again, $\qquad n = (5-\rho)/\rho^2\,(5\rho-1),$

therefore $\qquad dn/d\rho = (10\rho^2-76\rho+10)/\rho^3\,(5\rho-1)^2.$

From this we may infer that $dn/d\rho$ is always negative when ρ lies between 5 and $\frac{1}{5}$, and does not vanish for any value of ρ between these limits; therefore n and r increase together.

8022. (J. Brill, B.A.)—PQR is a triangle circumscribed to a parabola whose focus is S and vertex A. P′, Q′, R′ are the points of contact of QR, RP, PQ. SL, SM, SN are diameters of circles passing through S and touching the parabola at P′, Q′, R′ respectively, and SK is a diameter of the circles circumscribing the triangle PQR. O is the centroid of the triangle PQR, O′ that of the triangle P′Q′R′, and H that of the triangle LMN. QR, RP, PQ meet the tangent at the vertex in X, Y, Z. and U is a point taken in the same tangent, so that 3 . AU = AX + AY + AZ, Prove that AS . HK = 3 . SU . OO′.

Solution by G. G. Storr, M.A.; Rev. T. Galliers, M.A.; *and others.*



Let the coordinates of P′ be $(am_1^2,\ 2am)$ and the equation to the parabola $y^2 = 4ax$. Draw SL cutting the normal P′G at P′ in C, making $\angle P'SC = \angle SP'C$ and CL = CS; then SL is a diameter of the circle touching the curve in P′ and passing through S, and

$$\angle LSl = \pi - \theta - 2\theta = \pi - 3\theta.$$

Hence the coordinates of L are

$$x = Al = AS + 2CS\cos(\pi - 3\theta) = a - SP'\sec\theta\cos 3\theta = 3a\tan^2\theta = 3am_1^2,$$

$$y = Ll = 2CS\sin 3\theta = a(3m_1 - m_1^3),$$

with similar expressions for M and N; the coordinates of Q′ and R′ being $(am_2^2,\ 2am_2)$, $(am_3^2,\ 2am_3)$. If the equation to the tangent at P′ be $m_1 y = x + am_1^2$, the coordinates of P, Q, R are

$$-\left\{am_2m_3,\ a(m_2+m_3)\right\},\quad \left\{am_3m_1,\ a(m_3+m_1)\right\},\quad \left\{am_1m_2,\ a(m_1+m_2)\right\};$$

hence the coordinates of O, O′, H are

$$\left\{\tfrac{1}{3}a\Sigma(m_1m_2),\ \tfrac{2}{3}a\Sigma(m)\right\},\quad \left\{\tfrac{1}{3}a\Sigma(m^2),\ \tfrac{2}{3}a\Sigma(m)\right\},$$

$$\left\{a\Sigma(m^2),\ a\left[\Sigma(m) - \tfrac{1}{3}\Sigma(m^3)\right]\right\}.$$

Putting $x = 0$ in the equations of the tangents at P′, Q′, R′, we have $AX = am_1$, $AY = am_2$, $AZ = am_3$, hence $AU = \frac{1}{3}a\Sigma(m)$. Again, the equation to the circumcircle of the ΔPQR is

$$x^2 + y^2 - \left\{\Sigma(m_1m_2) + 1\right\}ax - \left\{\Sigma(m) - m_1m_2m_3\right\}ay + a^2\Sigma(m_1m_2) = 0.$$

This circle passes through S, and the coordinates of its centre are

$$\tfrac{1}{2}a\left\{\Sigma(m_1m_2) + 1\right\},\quad \tfrac{1}{2}a\left\{\Sigma(m) - m_1m_2m_3\right\},$$

The coordinates of K are easily found to be

$$a\Sigma(m_1m_2),\quad a\left\{\Sigma(m) - m_1m_2m_3\right\},$$

$$\therefore\quad SA . HK = a^2\left[\left\{\Sigma(m^2) - \Sigma(m_1m_2)\right\}^2 + \tfrac{1}{9}\left\{\Sigma(m^3) - 3m_1m_2m_3\right\}^2\right]^{\frac{1}{2}},$$

and $$SU . OO' = \tfrac{1}{3}a^2\left\{1 + \tfrac{1}{9}\left[\Sigma(m)\right]^2\right\}^{\frac{1}{2}}\left\{\Sigma(m^2) - \Sigma(m_1m_2)\right\}.$$

But $$\Sigma(m^3) - 3m_1m_2m_3 = (\Sigma m)\left\{\Sigma(m^2) - \Sigma(m_1m_2)\right\};$$

hence $$SA . HK = 3 . SU . OO'.$$

2555. (The late Professor De Morgan.)—The following is a theorem of which an elementary proof is desired. It was known before I gave it in a totally different form in a communication (April, 1867) to the Mathematical Society on the *conic octogram*; and the present form is as distinct from the other two as they are from one another. If I, II, III, IV be the consecutive chord-lines of one tetragon inscribed in a conic, and 1, 2, 3, 4 of another; the eight points of intersection of I with 2 and 4, II with 1 and 3, III with 2 and 4, IV with 1 and 3, lie in one conic section. A proof is especially asked for when the first conic is a pair of straight lines. There is, of course, another set of eight points in another conic, when the pairs 13, 24 are interchanged in the enunciation.

Solution by W. S. Foster.

Let the equations of the lines be

$$\text{I}, \alpha = 0;\ \text{II}, \beta = 0;\ \text{III}, \gamma = 0;\ \text{IV}, \lambda\alpha + \mu\beta + \nu\gamma = 0,$$

$$(1)\ l_1\alpha + m_1\beta + n_1\gamma = 0;\quad (2)\ l_2\alpha + \ldots = 0;\quad (3)\ l_3\alpha + \ldots = 0;$$

$$(4)\ l_4\alpha + \ldots = 0.$$

The conic circumscribing the quadrilateral of which I, II, III, IV are the consecutive sides is $\beta(\lambda\alpha + \mu\beta + \nu\gamma) = k_1\alpha\gamma$,

and the conic circumscribing the other quadrilateral is

$$(l_1\alpha + m_1\beta + n_1\gamma)(l_3\alpha + m_3\beta + n_3\gamma) = k_2(l_2\alpha + m_2\beta + n_2\gamma)(l_4\alpha + m_4\beta + n_4\gamma).$$

If the two quadrilaterals are inscribed in the same conic, the above two conics must coincide, and we shall have

$$\left.\begin{array}{ll} l_1l_3 - k_2l_2l_4 = 0, & l_1m_3 + l_3m_1 - k_2(l_2m_4 + l_4m_2) = r_1\lambda \\ m_1m_3 - k_2m_2m_4 = r_1\mu, & m_1n_3 + m_3n_1 - k_2(m_2n_4 + m_4n_2) = r_1\nu \\ n_1n_3 - k_2n_2n_4 = 0, & n_1l_3 + n_3l_1 - k_2(n_2l_4 + n_4l_2) = -r_1k_1 \end{array}\right\} \text{(A)}.$$

Therefore the relations that must hold between the constants will be given by the equations resulting from the elimination of k_1, k_2, r_1 from (A). Now, the eight points of intersection are (I, 2), (III, 2), (III, 4), (I, 4), (II, 3), (IV, 3), (IV, 1), (II, 1); and the first four form a quadrilateral, whose consecutive sides are 2, III, 4, I; and the second four, 3, IV, 1, II. And, if these are both inscribed in the same conic, the two equations

$$(l_2\alpha + m_2\beta + n_2\gamma)(l_4\alpha + m_4\beta + n_4\gamma) = k_3\alpha\gamma,$$

and $$(l_1\alpha + m_1\beta + n_1\gamma)(l_3\alpha + m_3\beta + n_3\gamma) = k_4\beta(\lambda\alpha + \mu\beta + \nu\gamma)$$

must coincide; therefore

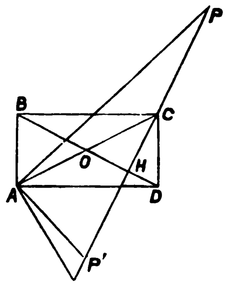
$$\left.\begin{array}{ll} l_1l_3 = r_2l_2l_4, & l_1m_3 + l_3m_1 - k_4\lambda = r_2(l_2m_4 + l_4m_2) \\ m_1m_3 - k_4\mu = r_2m_2m_4, & m_1n_3 + m_3n_1 - k_4\nu = r_2(m_2n_4 + m_4n_2) \\ n_1n_3 = r_2n_2n_4, & n_1l_3 + n_3l_1 = r_2(l_4n_2 + l_2n_4 - k_3) \end{array}\right\} \text{(B)}.$$

And, eliminating k_3, k_4, r_2 from (B), we shall find the same relations as those given by (A). Hence, if the first eight points lie on a conic, the second eight will also lie on a conic.

If the first conic is two straight lines, say α, γ, we must have m_1, n_1, l_3, m_3 each $= 0$, and the two conics will coincide.

9745. (Professor DE LONGCHAMPS.) — Soient A, C deux sommets opposés d'un rectangle ABCD : démontrer que (1) la perpendiculaire élevée de C sur BD rencontre la bissectrice de BAD en P ; démontrer que CP = CA ; et déduire de là (2) que la développée de l'hypocycloide à quatre rebroussements est aussi une hypocycloide à quatre rebroussements.

Solution by Professors IGNACIO BEYENS ; MATZ ; *and others.*



1. D'après la figure nous avons :

$$\text{CAP} = \text{BAC} - 45^\circ,\ \text{HCA} = 90^\circ - \text{COD} \ldots (1);$$

$$\text{APC} = \text{HCA} - \text{CAP} = 90^\circ - \text{COD} - (\text{BAC} - 45^\circ),$$

$$\text{APC} = 135^\circ - (\text{COD} + \text{BAC}),$$

$$\text{COD} + \text{BAC} = \text{BOA} + \text{BAC} = 180^\circ - \text{ABO},$$

$$\text{APC} = 135^\circ - (180^\circ - \text{ABO}) = \text{ABO} - 45^\circ \ldots (2).$$

D'après (1), (2), CAP = APC ; d'où CP = CA.

2. Si BD se meut sur les droites AB, AD, qui forment l'angle droit BAD, de telle manière que les extrémités B, D parcourent les côtés AB, AD, elle engendrera une hypocycloide à quatre rebroussements. Le centre *instantaneous* de rotation de BD dans ce mouvement sera C ; et, par suite, la perpendiculaire (PP′) menée par ce point à BD sera la normale qui passera par le centre de courbure de l'hypocycloide C. Ainsi, l'enveloppe de PP′ sera la développée de l'hypocycloide à quatre rebroussements.

Si nous menons la perpendiculaire AP′ sur AP, on aura dans doutes les positions de PP′, CP = CA, et par suite CP′ = CP = CA, l'angle PAP′ étant droit ; mais la droite AP, bissectrice de l'angle BAD, est fixe ; la perpendiculaire AP′ l'est aussi ; donc la droite PP′ = 2AC se meut sur les côtés de l'angle droit PAP′, ayant une longueur constante, et par conséquent l'enveloppe des droites est une hypocycloide à quatre rebroussements.

9831. (Professor MALET, F.R.S.)—Let p_1, p_2, p_3, p_4 be the perpendiculars from the centre of the quadric $a^{-2}x^2 + b^{-2}y^2 + c^{-2}z^2 - 1 = 0$ on the faces of a self-conjugate tetrahedron, a_1, a_2, a_3, a_4 the areas of the faces of the tetrahedron, and V its volume. If W be the volume of the tetrahedron formed by joining the feet of the perpendiculars, then

$$4a_1a_2a_3a_4\text{W} = 9p_1p_2p_3p_4\text{V}^3(a^{-2} + b^{-2} + c^{-2}).$$

Solution by the PROPOSER.

Let the equations of the sides of the self-conjugate tetrahedron be

$$\alpha_1 \equiv x\cos a_1 + y\cos\beta_1 + z\cos\gamma_1 - p_1 = 0,\quad \alpha_3 \equiv x\cos a_3 + y\cos\beta_3 + z\cos\gamma_3 - p_3 = 0,$$

$$\alpha_2 \equiv x\cos a_2 + y\cos\beta_2 + z\cos\gamma_2 - p_2 = 0,\quad \alpha_4 \equiv x\cos a_4 + y\cos\beta_4 + z\cos\gamma_4 - p_4 = 0.$$

Then, identifying the equations

$$a^{-2}x^2 + b^{-2}y^2 + c^{-2}z^2 - 1 = 0,\quad l\alpha_1^2 + m\alpha_2^2 + n\alpha_3^2 + p\alpha_4^2 = 0,$$

we have $$lp_1 \cos\alpha_1 + mp_2 \cos\alpha_2 + np_3 \cos\alpha_3 + pp_4 \cos\alpha_4 = 0,$$
$$lp_1 \cos\beta_1 + mp_2 \cos\beta_2 + np_3 \cos\beta_3 + pp_4 \cos\beta_4 = 0,$$
$$lp_1 \cos\gamma_1 + mp_2 \cos\gamma_2 + np_3 \cos\gamma_3 + pp_4 \cos\gamma_4 = 0,$$
$$l(1+kp_1^2) + m(1+kp_2^2) + n(1+kp_3^2) + p(1+kp_4^2) = 0,$$
where $$k = a^{-2} + b^{-2} + c^{-2}.$$
Hence, eliminating l, m, n, p, we have

$$\begin{vmatrix} p_1\cos\alpha_1, & p_2\cos\alpha_2, & p_3\cos\alpha_3, & p_4\cos\alpha_4 \\ p_1\cos\beta_1, & p_2\cos\beta_2, & p_3\cos\beta_3, & p_4\cos\beta_4 \\ p_1\cos\gamma_1, & p_2\cos\gamma_2, & p_3\cos\gamma_3, & p_4\cos\gamma_4 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

$$= -(a^{-2}+b^{-2}+c^{-2})\,p_1p_2p_3p_4 \begin{vmatrix} \cos\alpha_1, & \cos\alpha_2, & \cos\alpha_3, & \cos\alpha_4 \\ \cos\beta_1, & \cos\beta_2, & \cos\beta_3, & \cos\beta_4 \\ \cos\gamma_1, & \cos\gamma_2, & \cos\gamma_3, & \cos\gamma_4 \\ p_1, & p_2, & p_3, & p_4 \end{vmatrix},$$

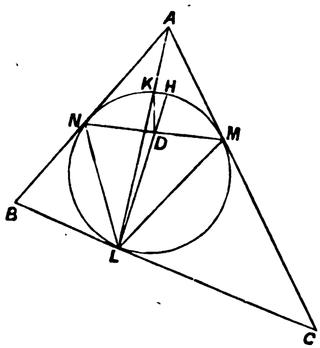
i.e., $$6W = (a^{-2}+b^{-2}+c^{-2})\,p_1p_2p_3p_4 \left\{a_1p_1 + a_2p_2 + a_3p_3 + a_4p_4\right\} \frac{9V^2}{2a_1a_2a_3a_4},$$
Hence the result.

9959. (E. M. Langley, M.A.)—Prove geometrically, without using transversals, that the lines joining the points of contact of the in-circle with the sides to the opposite angles are concurrent.

Solution by the Proposer; Professor Steggall; *and others.*

Let L, M, N be the points of contact of the in-circle with BC, CA, AB; let D be the mid-point of MN, and LD, AL meet the in-circle in H, K. Then it is known that DL, DK are equally inclined to the diameter through D; therefore

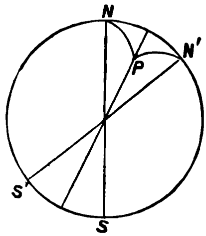
arc NK = arc HM.



But LH is the median through L of LMN; therefore LK is the symmedian. Similarly MB, NC are symmedians; therefore LA, MB, NC cointersect. [*Otherwise:* — Because A is the intersection of tangents at N, M, to the circle LMN; therefore the lines in question are the symmedian lines of the triangle LMN, and are therefore concurrent.]

9954. (F. R. J. Hervey.)—A celestial globe being fixed in any position, there are, at any instant, two opposite points of its surface, whose central vectors aim truly at the corresponding points of the celestial sphere. Show that, as the Earth rotates, the variable points of coincidence (as they may be called) describe great circles of their respective spheres.

Solution by Professors Schoute, Morel, *and others.*



As a figure on the sphere can be transferred from one position to another by a rotation round an axis through the centre, there are always two points of coincidence, the points where this axis cuts the sphere. When NS is the axis of the heavens, N′S′ that of the globe, and P a point of coincidence, the arcs NP and N′P are equal; therefore the locus of P is the great circle that bisects orthogonally the arcs NN′ and SS′.

9377. (R. Knowles, B.A.)—A circle PCD touches a rectangular hyperbola at P, and meets it again in CD; the circle of curvature at P meets it again in Q. Prove (1) that the poles of CD and PQ with respect to the hyperbola are on a line through its centre, parallel to the normal at P; (2) these chords are equally inclined to one of the asymptotes with the normal; (3) the distance of the mid-point of PQ from the centre of the curve is equal to ρ the radius of curvature; (4) if the circle touching at P has double contact, and r is its radius, r^3/ρ is equal to the square on the semi-axis.

Solution by Rev. T. Galliers, M.A.; *and* G. G. Storr, M.A.

1. Let (h, k) be the pole of CD Its equation is

$$kx+hy = 2a^2 \text{ and also } x'x+y'y-\lambda = 0 \quad \ldots\ldots\ldots\ldots (1, 2)$$

(Salmon's *Conics*, Art. 251); therefore $k/h = x'/y'$, and the line joining hk to the centre is parallel to the normal at P, of which the equation is

$$x'x-y'y-x'^2+y'^2 = 0 \ldots\ldots\ldots\ldots (3).$$

The proof is the same for PQ.

2. This is seen at once from (2) and (3).

3. From the equations to PQ, and the curve $x'x+y'y-x'^2-y'^2 = 0$, $xy = a^2$, we find the coordinates of its mid-point to be $(x'^2+y'^2)/2x'$, $(x'^2+y'^2)/2y'$, and therefore its distance from the centre

$$= (x'^2+y'^2)^{\frac{3}{2}}/2a^2 = \rho.$$

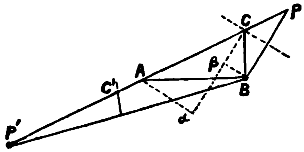
4. The equation to the circle of double contact is

$$xy-a^2 = \left\{lx+my-(lx'+my')\right\}^2,$$

with the condition $2lm = 1$, $l = m$; therefore $r^2 = x'^2+y'^2$ and $r^3/\rho = 2a^2$.

9842. (Professor Barbarin.)—Étant donnés deux points A, B, et une droite AX, trouver sur celle-ci un point C, tel que le produit des projections des droites CA, CB sur la bissectrice de l'angle ACB soit égal à un carré donné k^2.

Solution by Professors DeWachter, Chakravarti, *and others.*



If $C\alpha$, $C\beta$ be the projections of $AC = y$ and $CB = x$, c denoting AB,

$$C\alpha = y \cos \tfrac{1}{2}C \text{ and } C\beta = x \cos \tfrac{1}{2}C;$$

$$\therefore \quad C\alpha \cdot C\beta = xy \cos^2 \tfrac{1}{2}C = k^2.$$

Putting the value of $\cos \frac{1}{2}C$ in this expression, we get $(x+y)^2 = 4k^2 + c^2$.

Hence the following construction:—

On both sides of A take $AP = AP' = (4k^2+c^2)^{\frac{1}{2}}$ in the given line AX, and join BP, BP'. The perpendiculars erected in the middle points of BP and BP' will meet AX in the required points C, C'.

8741. (F. R. J. Hervey.) — Express as rational functions of the coefficients of the general equation of a central conic, referred to axes inclined at an angle ω, the functions $(r_1r_2)^2$ and $\tan(\theta_1+\theta_2-\omega)$ of the polar coordinates of the foci.

Solution by the Proposer; G. G. Storr, M.A.; *and others.*

The following investigation is founded upon the fact that the harmonic mean of the segments of a focal chord is constant for all directions of the chord. Let $ax^2+2bxy+cy^2+2dx+2ey+f=0$ ....(1)

be the given equation. Suppose $x = x'+mr$, $y = y'+nr$, where m and n are connected by the relation $m^2+2mn\cos\omega+n^2=1$ ....(2).

Substitute in (1), and let $Ar^2+2Br+C=0$ denote the resulting quadratic in r; its roots ρ_1, ρ_2 are therefore the segments of a chord drawn through (x', y') in a direction depending upon m, &c.

Suppose (x', y') to be a focus. Now, the segments having the above property are those which satisfy the polar equation $r(1+e\cos\theta)=l$, and are of *same* or *contrary* sign according as they are drawn from the focus in *contrary* or *same* directions. Hence $2\rho_1\rho_2/(\rho_1-\rho_2)$ is, apart from sign, the harmonic mean in question; and its square is $= C^2/(B^2-AC)$. The denominator (which alone depends upon m) has the form

$$Pm^2+2Qmn+Rn^2;$$

in which, suppressing accents and writing (h, k) for the centre of (1),

$$P = (b^2-ac)y^2+2(bd-ae)y+d^2-af = (b^2-ac)(y^2-2ky)+d^2-af,$$

$$Q = (ac-b^2)xy+(ae-bd)x+(cd-be)y+de-bf$$
$$= (b^2-ac)(-xy+kx+hy)+de-bf,$$

$$R = (b^2-ac)x^2+2(be-cd)x+e^2-cf = (b^2-ac)(x^2-2hx)+e^2-cf.$$

Our supposition requires that $Pm^2+2Qmn+Rn^2$ shall be a constant for all values of m and n which satisfy (2); which gives

$$P=R,\quad 2Q=(P+R)\cos\omega \qquad\qquad (3),$$

two equations between x and y which the coordinates of either focus must satisfy.

Let (x_1, y_1) and (x_2, y_2) be the foci. Putting either of these for x, y, in (3), and at the same time x_1+x_2 and y_1+y_2 for $2h$ and $2k$ respectively, we get
$$(b^2-ac)(y_1y_2-x_1x_2)=d^2-af-(e^2-cf),$$
$$(b^2-ac)\left\{x_1y_2+x_2y_1+(x_1x_2+y_1y_2)\cos\omega\right\}$$
$$=(d^2-af+e^2-cf)\cos\omega-2(de-bf);$$
but $\quad x_1=r_1\sin(\omega-\theta_1)/\sin\omega,\quad y_1=r_1\sin\theta_1/\sin\omega$, &c.,

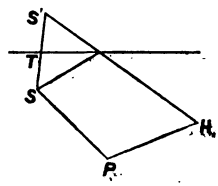
whence we find $\quad (y_1y_2-x_1x_2)\sin\omega=r_1r_2\sin(\theta_1+\theta_2-\omega)$,
$$x_1y_2+x_2y_1+(x_1x_2+y_1y_2)\cos\omega=r_1r_2\cos(\theta_1+\theta_2-\omega).$$
Let $S=\left\{d^2-af-(e^2-cf)\right\}\sin\omega,\ T=(d^2-af+e^2-cf)\cos\omega-2(de-bf)$;

therefore $\quad (r_1r_2)^2=(S^2+T^2)/(b^2-ac)^2,\quad \tan(\theta_1+\theta_2-\omega)=S/T.$

The following inferences seem worth noticing:—1. The equations (3) are those of two rectangular hyperbolas concentric with (1), and have only two real intersections. For the *parabola* they become the equations to two straight lines. Hence no point except a focus has the property in question. 2. The condition that the origin may be a focus is $S=T=0$. This follows directly from equations (3) and is true for any conic. 3. If the ratios $S:T:b^2-ac$ remain constant, and one focus describes, under any conditions, a path L; the locus of the other is an inverse of L with respect to the origin, *reflected* with respect to a fixed line. 4. If the axes are tangents, S, and therefore $\sin(\theta_1+\theta_2-\omega)$, vanishes. Hence a proof that the angles made by tangents from an external point and those made by the focal distances of the point have the same bisectors.

9925. (W. J. Greenstreet, M.A. Extension of Quest. 9766.)—Find the loci of the centres, and the second focus, of conics having one focus common, passing through a fixed point and touching a given straight line.

Solution by W. E. Brunyate; G. G. Storr, M.A.; *and others.*



Taking S′ so that SS′ is perpendicular to, and bisected by, the given tangent, if P be the given point and H any position of the other focus, we have

HS′ = the major axis of the ellipse
= SP + PH;

thus $\qquad$ S′H − PH = SP,

or the locus is a hyperbola with foci P and S′, and major axis SP.

8767. (D. Edwardes.)—Prove that, if $\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix} = 0$, then

$$(ac-b^2)\begin{vmatrix} ax^2+2bxy+cy^2, & bx^2+2cxy+dy^2 \\ bx^2+2cxy+dy^2, & cx^2+2dxy+ey^2 \end{vmatrix} = \begin{vmatrix} ax+by, & bx+cy \\ bx+cy, & cx+dy \end{vmatrix}^2.$$

Solution by the Proposer.

$$\begin{vmatrix} ax^2+2bxy+cy^2, & \&c. \\ \&c., & \&c. \end{vmatrix} = \begin{vmatrix} x(ax+by)+y(bx+cy), & \&c. \\ \&c., & \&c. \end{vmatrix}$$

$$= x^2\begin{vmatrix} ax+by, & bx+cy \\ bx+cy, & cx+dy \end{vmatrix} + xy\begin{vmatrix} ax+by, & cx+dy \\ bx+cy, & dx+ey \end{vmatrix} + y^2\begin{vmatrix} bx+cy, & cx+dy \\ cx+dy, & dx+ey \end{vmatrix},$$

where the coefficient of xy is

$$x^2\begin{vmatrix} a, & c \\ b, & d \end{vmatrix} + xy\begin{vmatrix} a, & c \\ c, & e \end{vmatrix} + y^2\begin{vmatrix} b, & d \\ c, & e \end{vmatrix}.$$

Now, since $\begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix} = 0$, and the determinant is symmetrical, therefore $\begin{vmatrix} a, & b \\ b, & c \end{vmatrix}\begin{vmatrix} a, & c \\ c, & e \end{vmatrix} = \begin{vmatrix} a, & c \\ b, & d \end{vmatrix}^2$, $\begin{vmatrix} a, & b \\ b, & c \end{vmatrix}\begin{vmatrix} b, & d \\ c, & e \end{vmatrix} = \begin{vmatrix} a, & c \\ b, & d \end{vmatrix}\begin{vmatrix} b, & c \\ c, & d \end{vmatrix}$, &c., &c.

Hence
$$\begin{vmatrix} a, & b \\ b, & c \end{vmatrix}\left\{x^2\begin{vmatrix} a, & c \\ b, & d \end{vmatrix} + xy\begin{vmatrix} a, & c \\ c, & e \end{vmatrix} + y^2\begin{vmatrix} b, & d \\ c, & e \end{vmatrix}\right\}$$

$$= \begin{vmatrix} a, & c \\ b, & d \end{vmatrix}\left\{x^2\begin{vmatrix} a, & b \\ b, & c \end{vmatrix} + xy\begin{vmatrix} a, & b \\ c, & d \end{vmatrix} + y^2\begin{vmatrix} b, & c \\ c, & d \end{vmatrix}\right\}$$

$$= \begin{vmatrix} a, & c \\ b, & d \end{vmatrix}\begin{vmatrix} ax+by, & bx+cy \\ bx+cy, & cx+dy \end{vmatrix}.$$

In the same way we find

$$\begin{vmatrix} a, & b \\ b, & c \end{vmatrix}\begin{vmatrix} bx+cy, & cx+dy \\ cx+dy, & dx+ey \end{vmatrix} = \begin{vmatrix} b, & c \\ c, & d \end{vmatrix}\begin{vmatrix} ax+by, & bx+cy \\ bx+cy, & cx+dy \end{vmatrix},$$

and therefore
$$\begin{vmatrix} a, & b \\ b, & c \end{vmatrix}\begin{vmatrix} ax^2+2bxy+cy^2, & \&c. \\ \&c., & \&c. \end{vmatrix}$$

$$= \left\{\begin{vmatrix} a, & b \\ b, & c \end{vmatrix}x^2 + \begin{vmatrix} a, & b \\ c, & d \end{vmatrix}xy + \begin{vmatrix} b, & c \\ c, & d \end{vmatrix}y^2\right\}\begin{vmatrix} ax+by, & bx+cy \\ bx+cy, & cx+dy \end{vmatrix}$$

$$= \begin{vmatrix} ax+by, & bx+cy \\ bx+cy, & cx+dy \end{vmatrix}^2.$$

8467. (R. Knowles, B.A.)—Show that the sum of the series

$$\frac{x^2}{n} + \frac{3x^4}{n^2}\cdot\frac{1}{2} + \frac{5x^6}{n^3}\cdot\frac{1}{3} + \ldots \text{ ad inf. is } \frac{2x^2}{n-x^2} + \log\left(1-\frac{x^2}{n}\right).$$

Solution by Prof. Ignacio Beyens; C. E. Williams, M.A.; *and others.*

The general term $\dfrac{(2m-1)\,x^{2m}}{n^m}\cdot\dfrac{1}{m} = 2\left(\dfrac{x^2}{n}\right)^m - \dfrac{1}{m}\left(\dfrac{x^2}{n}\right)^m$; $\therefore$ &c.

9691. (J. Brill, M.A.)—In a case of plane steady motion of a perfect incompressible fluid under the action of a conservative system of forces, if a curve be drawn such that the direction of motion of the fluid at all points of it is constant, then the acceleration of the fluid at any point of this curve is in the direction of the tangent at that point to the curve.

Solution by the Proposer; Professor Bordage; *and others.*

Let α and β be the component accelerations at any point of the fluid, and let $\eta = p/\rho + V$. Then we have

$$\frac{\partial\eta}{\partial x} + \alpha = 0, \quad \text{and} \quad \frac{\partial\eta}{\partial y} + \beta = 0,$$

which shows that the acceleration at any point of the fluid is normal to that curve of the family $\eta =$ const. that passes through it. Now, let $\xi = \tan^{-1}(v/u)$, and we have

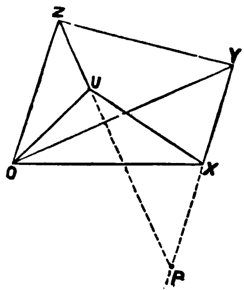
$$(u^2+v^2)\frac{\partial\xi}{\partial x} = u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \beta = -\frac{\partial\eta}{\partial y},$$

and $$(u^2+v^2)\frac{\partial\xi}{\partial y} = u\frac{\partial v}{\partial y} - v\frac{\partial u}{\partial y} = -\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\alpha = \frac{\partial\eta}{\partial x};$$

whence $\dfrac{\partial\xi}{\partial x}\dfrac{\partial\eta}{\partial x} + \dfrac{\partial\xi}{\partial y}\dfrac{\partial\eta}{\partial y} = 0$; consequently the two families of curves $\xi =$ const. and $\eta =$ const. are orthogonal, which proves the theorem.

9917. (E. M. Langley, M.A.)—A, B, C, D are four points on a circle. On the same circle any point O is taken. Show geometrically that the projections of O on the Simson-lines of the triangles BCD, CDA, DAB, ABC with respect to O lie in a straight line. Also, if this straight line be called the Simson-line of the quadrilateral ABCD with respect to O, and another point E be taken on the circle, the projections of O on the Simson-lines of the quadrilaterals BCDE, CDEA, DEAB, EABC, ABCD also lie in a straight line; and that the theorem can be extended.

Solution by Professor Steggall, M.A.; *the* Proposer; *and others.*



Let OX, OY, OZ, OU be perpendiculars on AB, BC, CD, DA. Then XY, YZ, ZU, UX are the pedal lines of the triangles ABC, BCD, CDA, DAB. And we have OY . OU = OX . OZ, whence the triangles XOY, ZOU are similar, and are situated as shown; whence the one triangle is the rotated and diminished representation of the other. Also the same statement holds of the triangles XOU, YOZ; and if XY, ZU meet in P, XPU = XOU, and the feet of the perpendicular from O on ZU, UX, XY are on the Simson-line of UXYP with respect to O. Hence, proceeding, the feet of the perpendiculars on ZU, UX, XY are in one line;

the feet of the perpendiculars on UX, XY, YZ are in one line. Thus the first result immediately follows, and the Simson-line of ABCD is the Simson-line with regard to the same point of the four triangles formed by the sides of the quadrilateral XYZU taken in order.

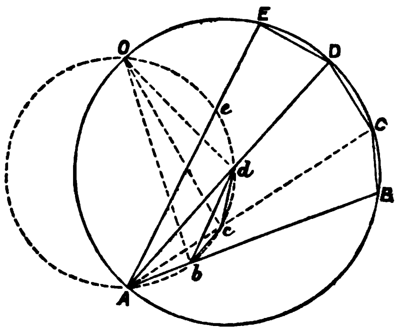
The second part is to be done immediately by analysis; the following proof includes the whole theorem:—

Let O be the origin, and let the coordinates of A, B, C, D ... be taken α, β, γ..., where α is the angle between OA and the diameter through O; and let d be the diameter. Then, if P, Q, R ... be the feet of the perpendiculars on AB, BC ... from O, the polar coordinates of P, Q, R ... are $d\cos\alpha\cos\beta \, . \, \alpha+\beta$, etc.; if X, Y, Z ... be the feet of the perpendiculars on PQ, QR, etc., their coordinates are $d\cos\alpha\cos\beta\cos\gamma$, $\alpha+\beta+\gamma$, etc.; and this is capable of obvious generalisation. As soon as the cycle is complete, we see that the points $(d\cos\alpha\cos\beta\cos\gamma\ldots\cos\mu,\ \alpha+\beta+\ldots+\mu)$, $(d\cos\beta\cos\gamma\ldots\cos\mu\cos\nu,\ d+\beta+\ldots+\mu+\nu)$, &c., lie on the line

$$r\cos(\theta-\alpha-\beta-\gamma\ldots-\nu) = d\cos\alpha\cos\beta\ldots\cos\nu,$$

the Simson-line of the n-sided figure.

[If b, c, d be the projections of O on AB, AC, AD, cd, db, bc are the Simson-lines of the triangles CDA, DAB, ABC; and b, c, d lie on the circle having OA for diameter; hence the projections of O on cd, db, bc are collinear. Similarly, the projections of O *on any three* of the four Simson-lines of the triangles BCD, CDA, DAB, ABC are collinear; hence all four are collinear. Again, if e be the projection of O on AE, the Simson-lines of the quadrilaterals ABCD, ACDE, ADEB, ABCE will, by the above proof, be the Simson-lines of the triangles



bcd, cde, deb, bec; and the projections of O *on these four* are also by that demonstration collinear (since O, b, c, d, e are concyclic). Similarly, the projections of O *on any four* of the Simson-lines of the quadrilaterals BCDE, CDEA, DEAB, EABC, ABCD are collinear. The same method is easily seen to be applicable to hexagons, heptagons, &c.]

9640. (J. O'Byrne Croke, M.A.)—In Question 9446, one of the two radii of the ellipsoid is supposed to lie always in the plane of yz, the other being at right angles to it, but otherwise free; suppose that, instead, two radii of an ellipsoid containing a right angle lie always on a plane passing through the axis of x, prove that a point on one of them, the square of whose distance from the centre is equal to the difference of

their squares, has for its locus the surface

$$\left\{\frac{(y^2+z^2)^2}{a^2}+x^2\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\right\}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$$
$$+(y^2+z^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=0.$$

Solution by the PROPOSER.

Let ρ_1, ρ_2 be radii of the ellipsoidal section through the axis of x containing a right angle, their direction angles being α, β, γ; α', β', γ'; and let ρ be a radius vector to a point in the locus. Then

$$\rho^2=\rho_1^2-\rho_2^2=\left(\frac{\cos^2\alpha}{a^2}+\frac{\cos^2\beta}{b^2}+\frac{\cos^2\gamma}{c^2}\right)^{-1}-\left(\frac{\cos^2\alpha'}{a^2}+\frac{\cos^2\beta'}{b^2}+\frac{\cos^2\gamma'}{c^2}\right)^{-1}\quad\text{..................(1).}$$

But $$\rho^2=x^2+y^2+z^2,$$

$$\cos^2\alpha=x^2/(x^2+y^2+z^2),\quad \cos^2\beta=y^2/(x^2+y^2+z^2),\quad \cos^2\gamma=z^2/(x^2+y^2+z^2);$$
$$\cos^2\alpha'=\sin^2\alpha,\qquad \cos^2\beta'=\cos^2\alpha\cos^2\beta\operatorname{cosec}^2\alpha,$$
$$\cos^2\gamma'=\cos^2\alpha\,(1-\cos^2\beta\operatorname{cosec}^2\alpha);$$

$$\left(\frac{\cos^2\alpha'}{a^2}+\frac{\cos^2\beta'}{b^2}+\frac{\cos^2\gamma'}{c^2}\right)^{-1}=\left\{\frac{\sin^2\alpha}{a^2}+\frac{\cos^2\alpha}{1-\cos^2\alpha}\left(\frac{\cos^2\beta}{b^2}+\frac{\cos^2\gamma}{c^2}\right)\right\}^{-1}$$
$$=\left\{\frac{\cos^2\beta+\cos^2\gamma}{a^2}+\frac{\cos^2\alpha}{\cos^2\beta+\cos^2\gamma}\left(\frac{\cos^2\beta}{b^2}+\frac{\cos^2\gamma}{c^2}\right)\right\}^{-1}.$$

Hence, by (1) $$x^2+y^2+z^2=(x^2+y^2+z^2)\Big/\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$$
$$-(y^2+z^2)(x^2+y^2+z^2)\Big/\left\{\frac{(y^2+z^2)^2}{a^2}+x^2\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\right\}.$$

$$\therefore\quad 1=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^{-1}-(y^2+z^2)\Big/\left\{\frac{(y^2+z^2)^2}{a^2}+x^2\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\right\};$$

therefore $$\left\{\frac{(y^2+z^2)^2}{a^2}+x^2\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)+y^2+z^2\right\}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)$$
$$=\frac{(y^2+z^2)^2}{a^2}+x^2\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right),$$

or $$\left\{\frac{(y^2+z^2)^2}{a^2}+x^2\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\right\}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$$
$$+(y^2+z^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=0,$$

the equation of the locus.

9896. (Professor STEGGALL, M.A.)—If $yz+bc=(xb-ya)(xc-za)$,

$$zx+ca=(yc-zb)(ya-xb),\quad xy+ab=(za-xc)(zb-yc),$$

then, prove that $$x^2+y^2+z^2=a^2+b^2+c^2=1.$$

Solution by Professor Wolstenholme; W. E. Brunyate; *and others.*

Writing the equations

$$\frac{yz}{bc}+1=a^2\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}-\frac{z}{c}\right),\ \text{\&c.},$$

we have $a^2+b^2+c^2 \equiv \left\{\left(\frac{yz}{bc}+1\right)\left(\frac{y}{b}-\frac{z}{c}\right)+\left(\frac{zx}{ca}+1\right)\left(\frac{z}{c}-\frac{x}{a}\right)\right.$

$$\left.+\left(\frac{xy}{ab}+1\right)\left(\frac{x}{a}-\frac{y}{b}\right)\right\}\Big/\left(\frac{y}{b}-\frac{z}{c}\right)\left(\frac{z}{c}-\frac{x}{a}\right)\left(\frac{x}{a}-\frac{y}{b}\right)=1.$$

Writing the equations

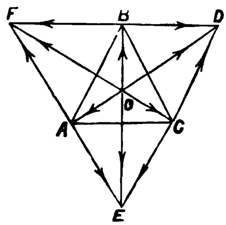
$$\frac{bc}{yz}+1=x^2\left(\frac{a}{x}-\frac{b}{y}\right)\left(\frac{a}{x}-\frac{c}{z}\right),\ \text{\&c.},$$

we see that $$x^2+y^2+z^2=a^2+b^2+c^2=1.$$

[This gives a proof without imaginaries of the well-known reciprocity of the orthogonal transformation in solid geometry.]

9849. (Professor Mukhopâdhyây, M.A.)—D, E, F are the vertices of equilateral triangles on the three sides of the triangle ABC; show that the centre of gravity of three equal particles placed at D, E, F coincides with the centroid of the triangle ABC.

Solution by Professor De Wachter; W. S. Foster; *and others.*



Let O be the centroid of ABC. If that point be also the centre of gravity of equal masses placed at D, E, F, or the centroid of triangle DEF, it is necessary that the forces represented both in magnitude and direction by OD, OE, OF shall be in equilibrium; which property admits of reciprocation. By decomposition of OD, OE, OF,

we get $2\,.\,\mathrm{OD}=\mathrm{OB}+\mathrm{BD}+\mathrm{OC}+\mathrm{CD}$;

$2\,.\,\mathrm{OE}=\mathrm{OC}+\mathrm{CE}+\mathrm{OA}+\mathrm{AE}$;

$2\,.\,\mathrm{OF}=\mathrm{OA}+\mathrm{AF}+\mathrm{OB}+\mathrm{BF}$;

therefore $2(\mathrm{OD}+\mathrm{OE}+\mathrm{OF})=2(\mathrm{OA}+\mathrm{OB}+\mathrm{OC})+(\mathrm{BD}+\mathrm{CD})$
$+(\mathrm{CE}+\mathrm{AE})+(\mathrm{AF}+\mathrm{BF})$.

Hence $\mathrm{OD}+\mathrm{OE}+\mathrm{OF}=d\mathrm{D}+e\mathrm{E}+f\mathrm{F}$; d, e, f being the middle points of the sides. The three forces dD, eE, fF meet in the circumcentre of ABC; they are proportional and perpendicular to the sides, therefore their resultant vanishes. And thus $\mathrm{OD}+\mathrm{OE}+\mathrm{OF}=0$, or the centre of gravity of the three equal masses at D, E, F coincides with the centroid of ABC.

[Let vector AB $=\alpha$, vector AC $=\beta$; then vector BC $=\beta-\alpha$, vector AD $=\alpha+(\beta-\alpha)(\cos\frac{1}{3}\pi+\epsilon\sin\frac{1}{3}\pi)$, and vector AE $=\beta(\cos\frac{1}{3}\pi-\epsilon\sin\frac{1}{3}\pi)$, vector AF $=\alpha(\cos\frac{1}{3}\pi+\epsilon\sin\frac{1}{3})$; hence vector from A to C.G. of DEF $=\frac{1}{3}$ (vector AD + vector AE + vector AF) $=\frac{1}{3}(\alpha+\beta)$ = vector from A to C.G. of ABC; therefore the centre of gravity of DEF coincides with that of ABC.]

9609. (Professor SYLVESTER, F.R.S.)—If ϕx is the number of proper fractions in their lowest terms none of whose denominators exceed the numerical quantity x; prove that $\phi x + \phi\frac{1}{2}x + \phi\frac{1}{3}x + \ldots = \frac{1}{2}[(\mathrm{E}x)^2 - \mathrm{E}x]$ (where as usual $\mathrm{E}x$ means x or the integral part of x, according as x is integer or fractional); and hence prove that, when x is infinite, $\phi x/x^2 = 3/\pi^2$, without making any assumption as to the form in which ϕx may be expressed as a function of x.

Solution by W. S. FOSTER.

The number of proper fractions, which have a denominator x, is equal to the number of numbers less than x and prime to it; and the number of those with denominator $\frac{1}{2}x$ is equal to the number of numbers less than x, and which have with x a G. C. M. = 2, and so on; and the number of numbers less than x, and having 2, 3, ... as their G. C. M. with x, must together be $x-1$; therefore, when x is a whole number,

$$\phi(x) + \phi(\tfrac{1}{2}x) + \phi(\tfrac{1}{3}x) + \ldots$$

must be equal to $\quad (x-1) + (x-2) + \ldots 2 + 1 = \frac{1}{2}x(x-1)$.

When x is not a whole number, the fractions of which $\phi(x)$ is the number will have for their denominators $\mathrm{E}x$, $\mathrm{E}x - 1$, ..., and, as before,

$$\phi(x) + \phi(\tfrac{1}{2}x) + \ldots = \tfrac{1}{2}[(\mathrm{E}x)^2 - \mathrm{E}x].$$

Let N_x be the number of numbers less than x, and prime to it, and let $a_1, a_2 \ldots a_x$ be all the prime numbers less than x; then, if $a_r, a_s \ldots$ be the prime divisors of x, and $a_u, a_v \ldots$ those of $x-1$,

$$\mathrm{N}_x = x - \frac{x}{a_r} - \frac{x}{a_s} - \ldots + \frac{x}{a_r a_s} + \ldots,$$

$$\mathrm{N}_{x-1} = (x-1) - \frac{x-1}{a_u} - \frac{x-1}{a_v} - \ldots + \frac{x-1}{a_u a_v} + \ldots \text{ and so on;}$$

and the numbers divisible by a_r will be $a_r, 2a_r, 3a_r \ldots \left(\mathrm{E}\dfrac{x}{a_r}\right)a_r$, therefore

$$\phi(x) = x + (x-1) + (x-2) + \ldots - \Sigma\frac{1}{a_r}\left[a_r + 2a_r + \ldots \left(\mathrm{E}\frac{x}{a_r}\right)a_r\right]$$
$$+ \Sigma\frac{1}{a_r a_s}\left[a_r a_s + 2a_r a_s + \ldots + \left(\mathrm{E}\frac{x}{a_r a_s}\right)a_r a_s\right] + \ldots$$

(Σ extending over all the prime numbers $a_1 \ldots a_x$),

$$\therefore\ \phi(x) = \frac{x(x+1)}{2} - \tfrac{1}{2}\Sigma\mathrm{E}\frac{x}{a_r}\left(\mathrm{E}\frac{x}{a_r} + 1\right) + \tfrac{1}{2}\Sigma\mathrm{E}\frac{x}{a_r a_s}\left(\mathrm{E}\frac{x}{a_r a^s} + 1\right) - \ldots,$$

and, when x is infinite, $\quad \dfrac{1}{x^2}\left(\mathrm{E}\dfrac{x}{a_r}\right)^2 = \dfrac{1}{a_r^2}$,

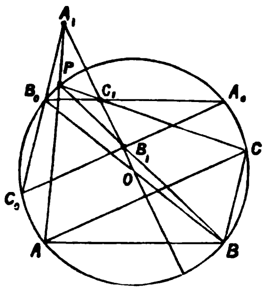
and then $\dfrac{\phi(x)}{x^2} = \dfrac{1}{2}\left(1 - \dfrac{1}{a_r^2} + \Sigma\dfrac{1}{a_r^2 a_s^2} - \ldots\right) = \dfrac{1}{2}\left(1 - \dfrac{1}{a_1^2}\right)\left(1 - \dfrac{1}{a_2^2}\right)\ldots,$

therefore $$\frac{\phi(x/r)}{x^2} = \frac{1}{r^2}\phi(x),$$

therefore $\quad \dfrac{\phi(x)}{x^2}\left(1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots\right) = \dfrac{1}{2}$, or $\phi(x)/x^2 = 3/\pi^2$.

9944. (Professor MOREL.)—**Un diamètre quelconque du cercle circonscrit à un triangle ABC coupe les côtés BC, CA, AB en A', B', C'; soient A_1, B_1, C_1 les symétriques de A', B', C' par rapport au centre O du cercle. Démontrer que les droites AA_1, BB_1, CC_1 sont concourantes.**

Solution by Professor SCHOUTE.



Soient A_0, B_0, C_0 les points diamétralement opposés à ABC dans le cercle (O), de manière que A_1, B_1, C_1 sont les points où le diamètre rencontre les côtés du triangle $A_0B_0C_0$. Soit P le point commun à BB_1 et CC_1. Le lieu de P est une conique, les deux faisceaux BP et CP étant rapportés projectivement l'un à l'autre; cette conique passe par A_0, B_0, C_0 et par B et C, de manière qu'elle coincide avec le cercle (O).

Comme BB_1 et CC_1 se coupent sur le cercle (O) il en est de même avec CC_1 et AA_1. Ainsi AA_1, BB_1, CC_1 passent par un même point du cercle circonscrit à ABC.

9961. (J. VILLADEMOROS.)—Trouver un nombre entier qui soit égal à la somme des chiffres de son cube.

Solution by R. W. D. CHRISTIE; G. G. STORR, M.A.; *and others.*

Let $$N^3 = (9x+y)^3 = 10^3a + 10^2b + 10c + d;$$
then $$N^3 - N = (N-1)(N)(N+1) = 9\,(111a + 11b + c);$$
thus we get three sets of possible values for N, viz.,

8, 17, 26, 35, 44, &c., 18, 27, 36, 45, 54, &c., 10, 19, &c., inadmissible.

Now, it is easy to prove that every cube is of one of the forms 9M, $9M \pm 1$. Thus the only values admissible for y are 8 and zero, and therefore the last row is altogether inadmissible.

In order to determine which of these numbers must be taken, we have $N = a+b+c+d$, where the letters are each < 10. Take 26, *e.g.*, where d is evidently 6. Thus $a+b+c = 20$, which is possible, therefore 26 is *one* number. Again, take $44 = N = a+b+c+d+e$ and $e = 4$. Thus $40 = a+b+c+d$, which is clearly impossible. A few of the desired Nos. are 8, 17, 26, 18, 27, &c.

7735. (R. KNOWLES, B.A.)—A circle passes through the ends of a chord of a parabola and its pole; prove that, if the chord passes through a given point on the axis, (1) the envelope of the polar of the vertex with

respect to the circle is an hyperbola, (2) the locus of the pole of this polar with respect to the parabola is an ellipse.

Solution by G. G. Storr, M.A.; Rev. T. Galliers, M.A.; *and others.*

The polar of (h, k) is $ky = 2a(x+h)$. Put $y=0$, therefore $x = -h$, a constant by the question. The equation of the circle is (see Vol. 41, p. 78)

$$x^2+y^2-(1/a)(k^2+2a^2)x-(k/a)(a-h)y+h(2a-h)=0,$$

and the equation of the polar of the vertex with respect to this circle is

$$(k^2+2a^2)x+k(a-h)y-2ah(2a-h)=0 \ldots\ldots\ldots\ldots\ldots(1).$$

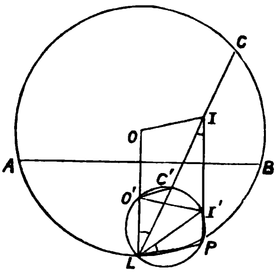
The envelope of this polar is the condition that this quadratic in k should have equal roots, or $8a^2x^2-(a-h)^2y^2-8ax(2a-h)x=0$, an hyperbola passing through the vertex.

Again, let (h_1, k_1) be the pole of (1) with respect to the parabola, then, making (1) identical with $k_1y = 2a(x+h_1)$, eliminating k from the resulting equations, and writing x and y for h_1 and k_1, we have as equation to the locus of the pole of (1) with respect to the parabola

$$2a^2(a-h)^2x^2+h^2(2a-h)^2y^2+2ah(2a-h)(a-h)^2x=0,$$

an ellipse passing through the vertex of the parabola.

Proof of Feuerbach's Theorem. *By* W. S. McCay, M.A.



Let O, I be the circumcentre and incentre of ABC; L the lowest point of the circumcircle; O′, I′ the reflexions of O, I below AB.

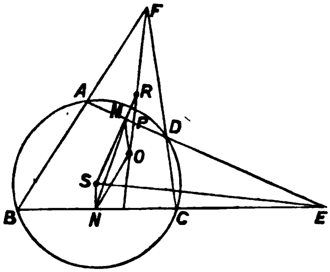
Describe a circle through LO′I′ cutting II′ in P and CIL in C′. Then O′C′ is equal and parallel to 2NI, where N is the nine-point centre, and its magnitude is $R-2r$.

The figure OP is a parallelogram by symmetry, and $IP = R$, $PI' = R-2r$, $LP = OI = D$. From the relation $D^2 = R(R-2r)$ we see that the triangles ILP, LI′P are similar; hence PLI′ = PIL = O′LC′, therefore $O'C' = PI' = R-2r$. Also IC′. IL = IP . II′ (by the small circle) $= 2Rr =$ IC . IL, therefore IC′ = IC. I is then the middle point of CC′, and N the middle point of CO′ (for, if H be the orthocentre, COO′H is a parallelogram, and N is middle point of OH). We see that $NI = \frac{1}{2}O'C' = \frac{1}{2}R-r$, and therefore the circles touch.

9910. (W. P. Casey.)—ABCD is a quadrilateral inscribed in a circle. The opposite sides meet in F, E; and the diagonals AC, BD intersect in

O ; M, N are the mid-points of AD, BC. Prove FO a tangent to the circum-circle of ΔONM.

Solution by R. F. Davis, M.A.; Professor Schoute; *and others.*



Let S be the centre of the circle; AD, BC meet in E; BA, CD in F; OF, SM in R; AD, OR in P; then, since SMEN are concyclic, the angle MNE = MSE = complement of SRO, for SE perpendicular to OR = OPD = OMD + MOR

= ONC + MOR,

from the similar triangles BOC, DOA. Hence we have

MOR = MNE − ONC = ONM;

and OR (that is, OF) touches the circle about ONM.

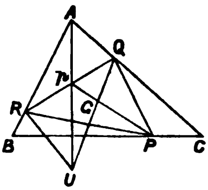
10053. (W. J. C. Sharp, M.A. Extension of Quest. 9513.)—If common tangents be drawn to a curve of the mth class and to a curve of the second class, and if these be arranged in m pairs, and from their m intersections other tangents be drawn to the first curve, prove that they will all touch a curve of class $m-2$. This theorem reduces to Quest. 9513, if the conic be replaced by two points, and if these points become coincident.

Solution by the Proposer.

This is the reciprocal of Question 8733—That if the $2m$ intersections of a conic and an m-ic curve be joined two and two, the m right lines so determined will cut the m-ic again in $m(m-2)$ points lying on an $(m-2)$-ic. Question 9513 follows when the conic consists of two coincident points and this introduces the idea of a curve which is the satellite of a point with respect to a given curve of class m. In the particular case of a class cubic, say a cuspidal cubic or tricuspidal quartic, it follows that, if three tangents be drawn from a point to the curve, the three tangents which can be drawn to the curve from the points of contact all pass through the same point. Similarly, if tangents be drawn from any point to a class quartic, nodal cubic say, the eight tangents which can be drawn from the points of contact touch the same conic.

9958. (R. H. W. Whapham.)—ABC is a given triangle, P any point in BC; find points Q and R in CA and AB respectively, such that the centroid of the triangle PQR may coincide with that of the triangle ABC.

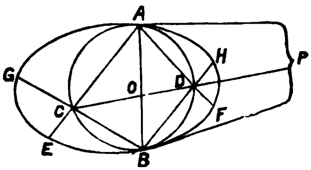
Solution by R. F. Davis, M.A.; J. C. St. Clair; *and others.*



Let G be the common centroid; produce PG to p so that $\text{PG} = 2\text{G}p$, then p is the mid-point of the side QR. Produce Ap to U so that $\text{A}p = p\text{U}$, parallels through U to CA, AB will then determine Q, R.

9941. (Professor Madhavarao.)—Two conics ACBD, GEFH have double contact at A and B. CD is the polar of a point in AB with regard to the first conic. If right lines ACE, ADF, BCG, BDH be drawn, show that the lines CD, EF, GH concur in a fixed point.

Solution by J. C. St. Clair; Rev. T. Galliers, M.A.; *and others.*



Since the pole of CD lies on AB, CD passes through P, the intersection of the tangents at A and B. Let AB, CD meet in O.

Then (A . PCOD) ≡ (A . PEBF) is an harmonic pencil, and since AP ≡ AA, (AEBF) is an harmonic system of points, and for similar reasons so also is (BGAH). Consequently EF, GH, CD are concurrent at the fixed point P. (Townsend, *Mod. Geom.*, Art. 257; Cremona, *Proj. Geom.*, Art. 195.)

9988. (Professor Hudson.)—A particle is projected with a given velocity in a medium in which the resistance varies as the cube of the velocity; find the time in which it will traverse a given distance, and the velocity which it will have at the end of a given time.

Solution by Professor De Wachter.

The equation of the problem is $\dfrac{d^2x}{dt^2} + k\left(\dfrac{dx}{dt}\right)^3 = 0$;

or, by putting $\dfrac{dx}{dt} = v$; $\quad \dfrac{dv}{dt} + kv^3 = 0.$

The first integration gives $2kt = \dfrac{1}{v^2} - \dfrac{1}{a^2}$ or $v^2 = \dfrac{a^2}{2a^2kt+1}$ ....(1),

where a is the given velocity at the common origin of time and distance.

Next, by replacing dx/dt in (1), we get $dx = \dfrac{dt}{\pm[2kt+1/a^2]}$; hence

$$kx = \pm[2kt+1/a^2]^{\frac{1}{2}} - 1/a,$$

or, finally, $$2at = 2x + akx^2 \text{ .... (2).}$$

The answer is given by (1) and (2).

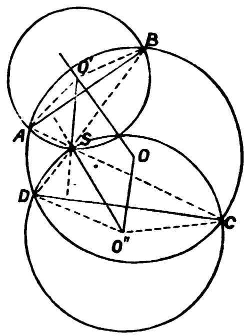
[The first result comes independently of the second, more naturally, thus:—

From $$\frac{d^2x}{dt^2} + k\left(\frac{dx}{dt}\right)^3 = 0, \quad -\frac{d^2x}{dt^2}\Big/\left(\frac{dx}{dt}\right)^2 = k\frac{dx}{dt},$$

$$\therefore \left(\frac{dx}{dt}\right)^{-1} = \frac{1}{a} + kx, \quad \therefore \frac{dt}{dx} = \frac{1}{a} + kx, \quad \therefore t = \frac{x}{a} + \frac{kx^2}{2}.\text{]}$$

9939. (Professor Fouché.)—On donne un cercle O, une corde fixe AB, et une corde CD de longueur constante, mais de position variable. On trace AC et BD, qui se coupent en S. Démontrer que le lieu du point S, et celui du centre du cercle circonscrit au triangle SCD, sont deux figures égales.

Solution by Professor De Wachter; R. F. Davis, M.A.; *and others.*



The complete locus of S is a circle described on AB, S being *within* or *without* the circumference O according to the intersection of AB and CD, being *without* or *within* that circumference.

Let O′ be the circumcentre of ASB and O″ that of CSD; O″ is at a given distance from O, and therefore, describes a circumference about it, with radius OO″. It is easy to see that, owing to the similarity of SAB and SDC, the quadrilaterals O′ASB and O″DSC must be similar figures.

Hence, the radii O′S and O″S are equally inclined on DSB and ASC, which implies (according to a known property) that O′S is perpendicular on CD and O″S on AB. Therefore OO′SO″ is a parallelogram, and O″O = O′S; thus O″ describes a circumference equal to the locus of S.

[Since both CD and the angle at S are constant, the circum-radius of the triangle SCD is also constant. Moreover, since CD, AB are anti-parallels with respect to the angle ASB, the line from S to the circum-centre of S coincides with the perpendicular from S on AB. Hence the loci of the question differ only in being shifted through a constant distance perpendicular to AB.]

9974. (V. Jamet.)—Intégrer l'équation aux dérivées partielles

$$\frac{dz}{dx}\,\frac{dz}{dy} = z\,\frac{d^2z}{dx\,dy}.$$

Solution by J. D. Hamilton Dickson, M.A.

Write $$\frac{1}{z}\,\frac{dz}{dx} = \frac{d\cdot}{dx}\left(\frac{dz}{dy}\right)\Big/\frac{dz}{dy},$$

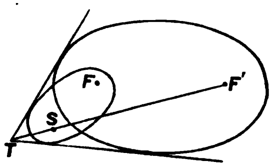
then $$\frac{d}{dx}\left(\log\frac{dz}{z\,dy}\right) = 0\,;$$

therefore $$\frac{dz}{z\,dy} = \text{const. independent of } x = \psi'(y).$$

Integrating again, $z = e^{\psi(y)}\,B$, where B is a constant independent of y, (say) $F(x)$. Hence the solution is $z = f(y)\,.\,F(x)$, where f and F are two arbitrary functions.

9918. (B. H. Steede, B.A.)—One fixed conic and another, given in all but position, have a common focus; prove that the locus of intersection of common tangents is a circle.

Solution by the Proposer; W. E. Brunyate; *and others.*



If both conics be central, the focus S which is not fixed describes a circle round the fixed common focus F, and will lie on the right line joining the remaining fixed focus F′ with the intersection T of the common tangents. Then, since product of the perpendiculars from the foci is equal to square of semi-axis minor, we see that the ratio of F′T to ST is same as ratio of the squares of semi-axes minor of the conics, and is therefore constant. Therefore the locus of T is a circle.

[If one conic be a parabola, supposing the focus F′ to be at an infinite distance, it is easy to prove that TS is constant, and that, therefore, the locus of T is still a circle.]

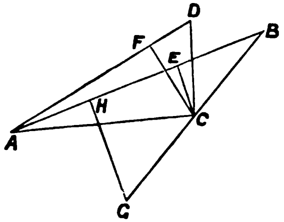
9639. (J. Young, M.A.)—Construct a quadrilateral whose diagonals AB, CD and one pair of opposite sides AD, BC are given in magnitude, such that the difference of the areas of the triangles ABC, ADC may be (1) equal to a given area, (2) a minimum. [See Vol. xxxv., p. 99, Quest. 6605, and Vol. xxxvii., p. 73, Quest. 6910.]

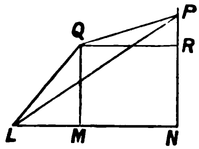
Solution by the PROPOSER; SARAH MARKS, B.Sc.; *and others.*

1. Take BG a fourth proportional to AD, AB, BC. Draw the perpendiculars CE, CF, GH. Twice the difference of the areas equals

$$AB \,.\, CE - AD \,.\, CF = AD\,(GH - CF);$$

also
$$AD^2 + DC^2 - 2AD \,.\, DF = AB^2 + BC^2 - 2AB \,.\, BE$$
$$= AB^2 + BC^2 - 2AD \,.\, BH.$$





Thus the problem reduces to:—Given the hypothenuses of two right-angled triangles BG, CD, and the differences of the sides GH − CF, BH − DF, to construct them. Draw the right-angled triangle PQR, so that PR = GH − CF, QR = BH − DF, and on PQ make the triangle PQL with PL = BG, QL = CD; then PLN, QLM are the triangles required.

2. When QR is given, and PR a minimum, PQ is also a minimum, and PQ is always greater than the difference of the given lines PL, QL, except when Q lies in the line PL. For this position the angle LPN = LQM, that is B = D, or the quadrilateral ABCD is cyclic.

9411. (ASPARAGUS.)—Two radii vectores OP, OQ of the curve $r = 2a\cos^3(\frac{1}{4}\pi + \frac{1}{3}\theta)$ are drawn equally inclined to the initial line; prove that the length of the intercepted arc is $a\alpha$, where α is the circular measure of the angle POQ.

Solution by the PROPOSER.

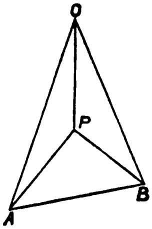
$$ds/d\theta = 2a\cos^2(\tfrac{1}{4}\pi + \tfrac{1}{3}\theta) = a\left\{1 + \cos(\tfrac{1}{2}\pi + \tfrac{2}{3}\theta)\right\} = a\,(1 - \sin\tfrac{2}{3}\theta);$$

whence s, that is the arc PQ $= a\int_{-\frac{1}{2}\alpha}^{\frac{1}{2}\alpha}(1 - \sin\tfrac{2}{3}\theta)\,d\theta = a\alpha.$

9761. (H. F. W. BURSTALL.)—Show that the potential of a uniform polyhedron of density σ at its centre is

$$\tfrac{1}{2}\sigma r^2\left\{\cot I \log\frac{1 + \cot I \cot \pi/n}{1 - \cot I \cot \pi/n} - 2\pi\,(1/n + 1/m - \tfrac{1}{2})\right\}.$$

Solution by the PROPOSER.



Let AB be an edge, O the centre, P the centre of the face; divide the pyramid into thin sections by planes parallel to the face.

$$\text{Potential} = \sigma \int_0^r dz \int_{-\pi/m}^{+\pi/m} d\theta \int_0^{z \cot I/\cos\theta} \frac{r\,dr\,d\theta}{(r^2+z^2)^{\frac{1}{2}}}$$

$$= \tfrac{1}{2}\sigma r^2 \left\{ \int_{-\pi/m}^{+\pi/m} \frac{d\theta\,(1-\sin^2 I \sin^2\theta)^{\frac{1}{2}}}{\sin I \sin\theta} - \frac{2\pi}{m} \right\},$$

and putting $\sin\theta \sin I = \sin\phi$, and noting that $\sin I = \dfrac{\cos \pi/n}{\sin \pi/m}$, we get

$$\tfrac{1}{2}\sigma r^2 \left\{ \int_{-(\pi/2-\pi/n)}^{+(\pi/2-\pi/n)} \frac{d\tan\phi}{\tan^2 I - \tan^2\phi} - 2\pi\,(1/m + 1/n - \tfrac{1}{2}) \right\},$$

and the result follows easily.

9897. (Professor GREINER.)—Par un point P, donné dans le plan du triangle ABC, on peut mener deux transversales telles que leurs points de rencontre A′, A″ avec BC soient les milieux des segments interceptés sur ces transversales par AB et AC. Démontrer que les points A′, A″, et les points analogues B′, B″, C′, C″ des côtés CA, AB sont sur une même conique.

Solution by Professor WOLSTENHOLME, M.A., Sc.D.

If a parabola (U) be drawn touching the sides of the triangle ABC, and touching BC in its middle point, then the intercept between AB, AC of any tangent to U will obviously be bisected by BC. Taking areal coordinates on ABC, let (X : Y : Z) be the point P, $lx+my+nz = 0$ one of the required transversals through P; then the equations for $l : m : n$ will be $lX + mY + nZ = 0$, $2mn = nl + lm$; so that $2mnX + (m+n)(mY+nZ) = 0$, so that the points A′, A″ are given by the equations

$$x = 0, \quad 2Xyz + (y-z)(Yz - Zy) = 0,$$

which lie on the conic through the six points A′, A″; B′, B″: C′, C″,

$$\frac{x^2}{X} + \frac{y^2}{Y} + \frac{z^2}{Z} - \frac{yz}{YZ}(2X+Y+Z) - \frac{zx}{ZX}(X+2Y+Z) - \frac{xy}{XY}(X+Y+2Z) = 0.$$

Writing this

$$(x+y+z)\left(\frac{x}{X} + \frac{y}{Y} + \frac{z}{Z}\right) = \frac{2(X+Y+Z)}{XYZ}(Xyz + Yzx + Zxy),$$

we see that it will be a circle if $X/a^2 = Y/b^2 = Z/c^2$; *i.e.*, if P be the Lemoine-point of the triangle ABC. If ρ be the radius of this circle,

$$\frac{\rho^2}{R^2} = \frac{1}{2} + \frac{a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2}{(a^2+b^2+c^2)^2},$$

which proves that $2\rho^2 > R^2$; and since also

$$\frac{\rho^2}{R^2} = \frac{3}{4} - \frac{3(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)}{4(a^2+b^2+c^2)^2},$$

$4\rho^2 < 3R^2$; so that ρ^2 lies always between $\frac{1}{2}R^2$ and $\frac{3}{4}R^2$. We may put this last equation in the form

$$\frac{4}{3}\frac{\rho^2}{R^2} = 1 - \frac{4\sin^2 A \sin^2 B \sin^2 C}{(\sin^2 A + \sin^2 B + \sin^2 C)^2} = 1 - \tan^2\omega,$$

where ω is the Brocard-angle of the triangle ABC.

The conic will be a parabola if $X^2+Y^2+Z^2-2YZ-2ZX-2XY=0$, *i.e.*, if P lie on the maximum ellipse inscribed in ABC.

The conic will be a rectangular hyperbola if

$$X+Y+Z=0, \text{ or } X(b^2+c^2-a^2)+Y(c^2+a^2-b^2)+Z(a^2+b^2-c^2)=0.$$

If $X+Y+Z=0$, three of the points A′, A″, &c. will be at infinity, and the conic degenerates into two straight lines (one at infinity). The other factor does give a rectangular hyperbola, for which P must lie on the radical axis of the circumcircle and nine points circle; or, if a, b, c be the middle points of the sides of the triangle ABC, P must lie on the same straight line with the points where the tangents to the nine points circle at a, b, c meet bc, ca, ab respectively.

The discriminant of the conic in general is

$$-6XYZ(X+Y+Z)(X^2+Y^2+Z^2+YZ+ZX+XY).$$

The conic obviously degenerates if $XYZ=0$, or $X+Y+Z=0$; and the value of the discriminant shows that it also degenerates if P lie on a certain conic homothetic with the maximum inscribed ellipse. This conic is, however, impossible. [If the transversal through P be divided by the sides in any other constant ratio than one of equality, the six points will not lie on one conic.]

9921. (D. Biddle.)—A random straight line cuts a given plane triangle. Prove that the average length of the portion which the triangle intercepts, is that of a quadrant of the in-circle ($= \frac{1}{2}\pi r$).

Solution by the Proposer; W. E. Brunyate; *and others.*

It is possible to draw a line, parallel to any random line which cuts a triangle, through one of the two sides so cut and the opposite angle; and the average of the intercepts of random lines parallel to such particular random line, will be half the intercept so drawn through the angle. The proportionate number of lines represented by the particular intercept thus drawn, is indicated by the sum of the perpendiculars upon it from the two other angles; and the product of half the intercept and this sum of the perpendiculars will give the *weight* attaching to it in the grand total from which the final average is drawn. But the product so obtained in every case $= \Delta$, the area of the triangle; therefore the grand total $= \Delta\pi/d\theta$, which must be divided by the total sum of the perpendiculars. The same

result is obtained if $\pi\Delta$ be divided by the following integral, formed by summing the perpendiculars, each multiplied by $d\theta$:

$$\int_0^A \{b \sin\theta + c\sin(A-\theta)\}\, d\theta + \int_0^B \{c\sin\theta + a\sin(B-\theta)\}\, d\theta$$
$$+ \int_0^C \{a\sin\theta + b\sin(C-\theta)\}\, d\theta$$
$$= (b+c)(1-\cos A) + (a+c)(1-\cos B) + (a+b)(1-\cos C) = a+b+c.$$

Therefore the average intercept $= \pi\Delta/(a+b+c)$. But, as is well known, $r = 2\Delta/(a+b+c)$. Hence the result given in the question.

[If the line fall in any given direction, the mean value for that direction will be Δ / (perp. dist. between extreme lines). The probability that the direction should lie within an angle $d\theta$ of this direction $\propto p\,d\theta$, where p is the greatest breadth in the perpendicular direction; thus the mean value is

$$\int \Delta p^{-1} p\theta \Big/ \int p\, d\theta = \pi\Delta \Big/ \int p\, d\theta.$$

But if the perpendicular direction fall within BAD (AD perpendicular to BC) p = BA cos (angle between BA and given direction); therefore $\int p\,d\theta = \mathrm{C}\int\cos\theta\, d\theta = \mathrm{C}\sin \mathrm{BAD} = \mathrm{BD}$ for the given limits, and therefore the value of $\int_0^\pi p\,d\theta$ is $a+b+c$. Thus the mean value

$$= \pi\Delta/(a+b+c) = \tfrac{1}{2}\pi\Delta/s = \tfrac{1}{2}\pi r.]$$

9932. (Professor HUDSON.)—If the same hyperbola be described by particles under the action of an attractive force to one focus and a repulsive force to the other, prove that the velocities are equal at the points for which the forces are equal.

Solution by Professor WOLSTENHOLME, M.A., Sc.D.

If S, S′ be the foci of the hyperbola, then, when the particle, under a force to S, is at a point P of the hyperbola, V^2/ρ = normal acceleration = F . SY/SP, mF being the force at P, SY perpendicular on the tangent at P, = F . BC/CD (with the usual notation), or $V^2/F = CD^2/AC = SP.S'P/AC$. So, if V′, F′ be the corresponding values at the same point P when the particle is describing the hyperbola under a force to S′, $V'^2/F' = SP.S'P/AC$; or $V^2 : V'^2 = F : F'$; and, when $V = V'$, F will $=$ F′.

[A very well known theorem in central orbits is that the space due to the velocity at any point is one-fourth the chord of curvature through the centre of force. Since the chords of curvature through the two foci are equal, this gives at once $V^2/F = V'^2/F'$.]

9943. (Professor BORDAGE.)—If a, b, c are the terms of rank m, n, p of (1) an arithmetic progression, (2) a geometric progression, prove that

$$a(n-p) + b(p-m) + c(m-n) = 0, \quad a^{n-p} \times b^{p-m} \times c^{m-n} = 1.$$

Solution by Isabel Maddison; Lucy Baker; *and others.*

1. $a = x + (m-1)\,a$, &c., therefore $b - c = (n-p)\,a$, &c., and

$$a\,\{a\,(n-p) + \&c.\} = \Sigma a\,(b-c) = 0.$$

2. $a = xa^{m-1}$, &c., $\therefore\ a^{n-p} = b/c$, $\therefore\ (n-p)\log a = \log b - \log c$, and $\Sigma\,(n-p)\log a = 0$; therefore, &c.

[If a, b, c are the m^{th}, n^{th}, p^{th} terms of the arithmetic progression also, the property $a\,(n-p) + b\,(p-m) + c\,(m-n) = 0$ is also true.]

9937. (Professor Déprez.)—La base AC d'un triangle est fixe, et l'angle au sommet P est constant. Démontrer que la droite qui joint les pieds des symédianes issues de A et C enveloppe une conique.

Solution by R. F. Davis, M.A.; Professor Beyens; *and others.*

Let P be the vertex of a triangle having a fixed base CA and a fixed vertical angle; CB, AB the tangents at the extremities of the base to the fixed circumcircle whose trilinear equation referred to ABC is $\gamma\alpha = \beta^2$. Then the equation to the tangent at P $(1 : \mu : \mu^2)$ being $\mu^2\alpha - 2\mu\beta + \gamma = 0$, this line meets BC, AB in points Q, R whose coordinates are $(0 : 1 : 2\mu)$ and $(2 : \mu : 0)$.

Hence, if AQ, CP meet in U $(1 : \mu : 2\mu^2)$ and AP, CR in V $(2 : \mu : \mu^2)$, the equation to UV (which is the line required) is $\mu^2\alpha - 3\mu\beta + \gamma = 0$; consequently its envelope is $9\beta^2 = 4\gamma\alpha$, which represents a conic having double contact with the circumcircle at C, A.

It is interesting to notice that the point of contact of this line with its envelope $(1 : \frac{3}{2}\mu : \mu^2)$, the Symmedian point of PCA $(1 : \frac{1}{2}\mu : \mu^2)$, the point-pair U, V all lie on different fixed conics of the family $\gamma\alpha = 2\beta^2$.

9931. (Professor Wolstenholme, M.A., Sc.D.)—Prove that

$$\int_0^\infty \frac{x^{m-1}\,dx}{(1+2x\cos\alpha+x^2)^m\,(1+x^n)} = \int_0^1 \frac{x^{m-1}\,dx}{(1+2x\cos\alpha+x^2)^m}$$

$$= \frac{1}{2^m\sin^{2m-1}\alpha}\int_0^\alpha (\cos x - \cos\alpha)^{m-1}\,dx = \frac{1}{2^m\,(m-1)\,!}\left\{\frac{1}{\sin\alpha}\,\frac{d}{d\alpha}\right\}^{m-1}\left(\frac{\alpha}{\sin\alpha}\right)$$

$$\int_0^\infty \frac{F\left(\dfrac{2x}{1+x^2}\right)}{1+x^n}\,\frac{dx}{x} = \int_0^{\frac{1}{2}\pi} F(\sin\theta)\,\frac{d\theta}{\sin\theta}.$$

Solution by D. Edwardes, B.A.; T. D. H. Dickson, M.A.; *and others.*

1. $$\mathrm{I} = \int_0^1 \frac{x^{m-1}\,dx}{(1+2x\cos\alpha+x^2)^m\,(1+x^n)} + \int_1^\infty \frac{x^{m-1}\,dx}{(1+2x\cos\alpha+x^2)^m\,(1+x^n)};$$

the 2nd integral becomes $\int_0^1 \frac{y^{m+n-1}\,dy}{(1+2y\cos\alpha+y^2)^m\,(1+y^n)}$, putting $\frac{1}{y}$ for x. Hence

$$I = \int_0^1 \frac{x^{m-1}\,dx}{(1+2x\cos\alpha+x^2)^m}\left(\frac{1}{1+x^n}+\frac{x^n}{1+x^n}\right) = \int_0^1 \frac{x^{m-1}\,dx}{(1+2x\cos\alpha+x^2)^m}$$

$$= \tfrac{1}{2}\int_0^\alpha (\sin\alpha\cot\tfrac{1}{2}\phi-\cos\alpha)^{m-1}\sin^{2m-2}\tfrac{1}{2}\phi\,\mathrm{cosec}^{2m-1}\alpha\,d\phi$$

$$\text{if}\quad x = (\sin\alpha\cot\tfrac{1}{2}\phi-\cos\alpha)^{-1}$$

$$= \frac{1}{2\sin^{2m-1}\alpha}\int_0^\alpha (\sin\alpha\cos\tfrac{1}{2}\phi\sin\tfrac{1}{2}\phi-\cos\alpha\sin^2\tfrac{1}{2}\phi)^{m-1}\,d\phi$$

$$= \frac{1}{2^m\sin^{2m-1}\alpha}\int_0^\alpha \left[\cos(\alpha-\phi)-\cos\alpha\right]^{m-1}d\phi$$

$$= \frac{1}{2^m\sin^{2m-1}\alpha}\int_0^\alpha (\cos\phi-\cos\alpha)^{m-1}\,d\phi = F(m), \text{ say.}$$

We easily get

$$\frac{d}{d\alpha}.F(m) = \frac{m^2}{(m-1)\,2^m\sin^{2m}\alpha}\int_0^\alpha (\cos x-\cos\alpha)^m\,dx$$

$$+\frac{(2m-1)\cos\alpha}{(m-1)\,2^m\sin^{2m}\alpha}\int_0^\alpha(\cos x-\cos\alpha)^{m-1}dx-\frac{1}{2^m\sin^{2m-2}\alpha}\int_0^\alpha(\cos x-\cos\alpha)^{m-2}dx$$

$$= \frac{m^2}{(m-1)\,2^m\sin^{2m}\alpha}\int_0^\alpha (\cos x-\cos\alpha)^m\,dx-\frac{1}{m-1}\frac{d}{d\alpha}.F(m),$$

or $$\frac{d}{d\alpha}.F(m) = \frac{m}{2^m\sin^{2m}\alpha}\int_0^\alpha (\cos x-\cos\alpha)^m\,dx,$$

therefore $$\frac{1}{\sin\alpha}\frac{d}{d\alpha}F(m) = 2m\,F(m+1);$$

and evidently $$F(1) = \frac{\alpha}{2\sin\alpha},$$

and therefore $$F(m) = \frac{1}{2^m\,\overline{m-1}\,!}\left(\frac{1}{\sin\alpha}\frac{d}{d\alpha}\right)^{m-1}\left(\frac{\alpha}{\sin\alpha}\right).$$

2. $$\int_0^\infty \frac{F\left(\frac{2x}{1+x^2}\right)}{1+x^n}\frac{dx}{x} = \int_0^{\frac{1}{2}\pi}\frac{F(\sin 2\theta)}{1+\tan^n\theta}.\frac{d\theta}{\sin\theta\cos\theta}\quad(x=\tan\theta)$$

$$= \int_0^{\frac{1}{2}\pi} F(\sin 2\theta)\left\{\frac{1}{1+\tan^n\theta}+\frac{1}{1+\cot^n\theta}\right\}.\frac{d\theta}{\sin\theta\cos\theta}$$

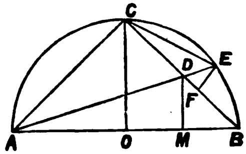
by grouping together the elements equally distant from the limits

$$= \int_0^{\frac{1}{2}\pi} F(\sin 2\theta)\frac{d\theta}{\sin\theta\cos\theta} = \int_0^{\frac{1}{2}\pi} F(\sin\phi)\frac{d\phi}{\sin\phi},\ \text{where } 2\theta=\phi.$$

9957. (W. J. Greenstreet, B.A.)—AB is the diameter, C the mid-point of the arc of a semicircle; D is the mid-point of the chord BC; let

AD be produced to meet the circle in E, EF is perpendicular to BC; show that CF = 3EF.

Solution by R. F. Davis, M.A.; R. H. W. Whapham, B.A.; *and others.*



Let O be the centre; draw DM perpendicular to OB; then OM = semi-radius = DM, and AM = 3DM; therefore, by similar triangles, CF = 3EF.

9966. (R. A. Roberts, M.A.)—Show that the focus of the cubic $y^3 - px^2 = 0$ is given by $27x = 8p\cos\omega$, $27y = 4p(1+2\cos 2\omega)$, where ω is the angle between the axes of coordinates.

Solution by Professor Wolstenholme, M.A., Sc.D.

Let $x + y\cos\omega + iy\sin\omega = m$ be a focal tangent to the cubic $y^3 = px^2$; then the equation $my^3 = px^2(x + y\cos\omega + iy\sin\omega)$ must have equal roots, the condition for which is $4p^3(\cos 3\omega + i\sin 3\omega) = 27mp^2$,

or $$27m = 4p(\cos 3\omega + i\sin 3\omega);$$

and the focus is given by the equation

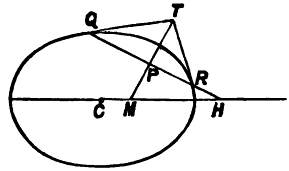
$$27(x + y\cos\omega + i\sin\omega) = 4p(\cos 3\omega + i\sin 3\omega),$$

i.e., $$27(x + y\cos\omega) = 4p\cos 3\omega, \quad 27y\sin\omega = 4p\sin 3\omega,$$

or $$27y = 4p(1 + 2\cos 2\omega), \quad 27x = -8p\cos\omega.$$

9861. (Rev. T. C. Simmons, M.A.)—If a chord of a central conic pass through a fixed point on the axis, prove that the locus of the foot of the perpendicular drawn on it from its pole is one of a system of circles having the other axis of the conic for their common radical axis; and give the corresponding theorem for the parabola.

Solution by the Proposer.



Let $(h, 0)$ be the coordinates of the fixed point H, taken on the major axis, and $(a^2/h, t)$ those of T, the pole of the chord HRQ. The equation of the perpendicular TPM is at once found to be

$$(t/b^2)(x - a^2/h) = (1/h)(y - t),$$

from which on putting $y = 0$, we obtain

$$\text{CM} = (a^2 - b^2)/h,$$

showing that M is fixed. Hence the locus of P is the semicircle on MH.

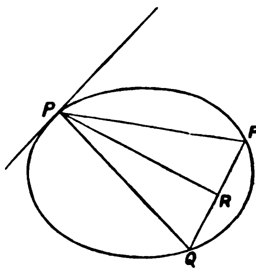
Also, since $CM.CH = a^2 - b^2 =$ constant for all positions of H, C is on the radical axis of all the circles, which is consequently the minor axis of the ellipse, the foci being the two limiting points. A similar result would have followed, had H been taken originally on the minor axis.

If the conic had been a parabola with vertex A and focus S, then we should have found, as before, the locus of P to be a circle; but in this case $AM + AH$ would have been equal to the constant 2AS, showing that all the circles had S for their common centre.

The connection between the two theorems for the ellipse and parabola is easily seen; as concentric circles may be regarded as coaxal circles whose radical axis is the line at infinity, and one of whose limiting points is the common centre.

9857. (W. J. Greenstreet, M.A.)—The normal to an ellipse at P meets the curve again in Q. P′, any point on the curve, is joined to P and Q. A perpendicular is drawn from P to QP′ cutting QP′ in R; find the locus of R as PP′ turns round P.

Solution by R. H. W. Whapham, B.A.; Rev. T. Galliers; *and others.*



Take the normal and tangent as axes of (x, y) respectively; then, the equations of the conic and PP′ being

$$ax^2 + 2hxy + by^2 + 2gx = 0, \quad y = mx,$$

at P′ we have

$$x = -\frac{2g}{a + 2hm + bm^2}, \quad y = -\frac{2mg}{a + 2hm + bm^2};$$

coordinates of Q are $(-2g/a, 0)$; equation of QP′ is $x + (2h + bm)\,y + 2g/a = 0$ ...... (1),

of PR is $-(2h + bm)\,x - y = 0$............ (2).

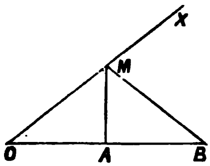
Hence, eliminating m from (1), (2), we get, for the locus of R, $ax^2 + ay^2 + 2gx = 0$, which represents a circle passing through P, Q, and whose centre is $-g/a, 0$; thus the locus of R is the circle on PQ as diameter.

[*Otherwise* :—Since the angle at R is always a right angle, and PQ is fixed; therefore the locus of R is a circle described on PQ as diameter.]

9847. (Professor Ignacio Beyens.)—Soient O, A, B trois points fixes sur une droite donnée par le point O, on mène une droite quelconque OX, et l'on détermine sur cette dernière le point M tel que l'angle AMB soit *maximum*. Si l'on fait la même construction pour toutes les droites

qui passent par O, lequel est le lieu géométrique des points M ainsi déterminés?

Solution by W. S. Foster; Professor Aiyar; *and others.*



When the angle AMB is a maximum, the circle described about AMB will touch OX in M; therefore $\qquad OM^2 = OA \cdot OB;$ hence the locus of M is a circle.

9906. (Professor Emmerich, Ph.D.)—Solve the equation

$$0 = \begin{vmatrix} \dfrac{\sin(x-\alpha_1)}{\sin x}, & \cos\alpha_2, & \cos\alpha_3, & \ldots\ldots & \cos\alpha_n \\ \cos\alpha_1, & \dfrac{\sin(x-\alpha_2)}{\sin x}, & \cos\alpha_3, & \ldots\ldots & \cos\alpha_n \\ \cos\alpha_1, & \cos\alpha_2, & \dfrac{\sin(x-\alpha_3)}{\sin x}, & \ldots\ldots & \cos\alpha_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \cos\alpha_1, & \cos\alpha_2, & \cos\alpha_3, & \ldots\ldots & \dfrac{\sin(x-\alpha_n)}{\sin x} \end{vmatrix}$$

Solution by Professor Wolstenholme; R. F. Davis, M.A.; *and others.*

By a known result (first given, I think, by Dr. Ferrers in the *Quarterly Journal*),

$$\begin{vmatrix} z_1, & a_2, & a_3 \ldots a_n \\ a_1, & z_2, & a_3 \ldots a_n \\ a_1, & a_2, & z_3 \ldots a_n \\ \ldots & \ldots & \ldots \\ a_1, & a_2, & a_3 \ldots z_n \end{vmatrix} \equiv (z_1-a_1)(z_2-a_2)\ldots(z_n-a_n)\left\{1+\frac{a_1}{z_1-a_1}+\ldots+\frac{a_n}{z_n-a_n}\right\}.$$

If in this identity we put $a_1 \equiv \cos\alpha_1$, $a_r = \cos\alpha_r$; $z_r = \sin(x-\alpha_r)/\sin x$, we have
$$z_r - a_r = -\sin\alpha_r \cot x,$$
and
$$a_r/(z_r-a_r) = -\cot\alpha_r \tan x;$$
so that the value of the proposed determinant is
$$(-1)^n \sin\alpha_1 \sin\alpha_2 \ldots \sin\alpha_n (\cot x)^n \left\{1-\tan x(\cot\alpha_1+\cot\alpha_2+\ldots+\cot\alpha_n)\right\},$$
and the equation in $\cot x$ has $n-1$ roots zero, and the finite root
$$= \cot\alpha_1 + \cot\alpha_2 + \ldots + \cot\alpha_n.$$

[The solution of this equation is $\cot x = \cot\alpha_1 + \ldots + \cot\alpha_n$; for from it we can obtain n equations of the type
$$(\cot\alpha_1 - \cot x) + \cot\alpha_2 + \ldots + \cot\alpha_n = 0,$$
or
$$\sin(x-\alpha_1)\operatorname{cosec} x \cdot \operatorname{cosec}\alpha_1 + \cos\alpha_2 \cdot \operatorname{cosec}\alpha_2 + \ldots = 0;$$
which, being linear and homogeneous in the variables $\operatorname{cosec}\alpha_1, \ldots$, give rise to an eliminant in the form of the determinant of the question.]

10025. (Professor SYLVESTER, F.R.S.)—P is a point on an ellipse, the circle of curvature of which cuts the ellipse in Q; another circle touching the conic at P, cuts the conic in two points R, S; another circle through QRS cuts the conic in a given point A. Show that there are five positions of P which satisfy this condition, and that they are the apices of a pentagon of maximum area that can be inscribed in the ellipse.

Solution by J. D. H. DICKSON, M.A.; Professor SCHOUTE, *and others.*

The tangent at P must obviously be parallel to QA, and PQ and RS are parallel. Hence a circle passing through P, A, Q touches the conic at Q. Let α, ϖ, κ be the eccentric angles of A, P, Q; then

$$\varpi + 2\kappa + \alpha = 6\pi \quad\text{.............................. (1).}$$

Also the equations of the tangent at P, and AQ, are

$$\frac{x}{a}\cos\varpi + \frac{y}{b}\sin\varpi = 1,$$

$$\frac{x}{a}\cos\frac{\kappa+\alpha}{2} + \frac{y}{b}\sin\frac{\kappa+\alpha}{2} = \cos\frac{\kappa-\alpha}{2};$$

whence, for parallelism, $\quad -2\varpi + \kappa + \alpha = 4\pi$ .... (2).

From (1) and (2), $\varpi = \frac{1}{5}(2\pi + \alpha)$, or, noting that 2π may be replaced by $2n\pi$, where $n = 1, 2, 3, 4, 5$, we get the five values of ϖ required. The rest follows obviously.

1916. (Sir R. BALL, LL.D., F.R.S.)—Show that the equation of squares of differences of the biquadratic $(a, b, c, d, e)(x, 1)^4 = 0$ has for its discriminant (H being $= b^2 - ac$, &c., as in Quest. 1876)

$$(27J^2 - I^3)^2 (4H^3 - a^2IH - a^3J)^2 (55296H^3J + 2304aH^2I^2 - 16632a^2HIJ - 625a^3I^3 - 9261a^2J^2)^2.$$

Solution by Professor SEBASTIAN SIRCOM.

The required discriminant expressed in terms of the roots of the original quartic will be $L^2M^2N^2$, where L is the product of six terms of the form $(\alpha-\beta)^2$ and is the discriminant of the quartic, M the product of three terms of the form $\alpha-\beta+\gamma-\delta$, so that $M = G = 4H^3 - a^2IH - a^3J$, and N is the product of twelve terms of the form $\alpha+\beta-2\gamma$, and is of the ninth order at most in any one. Then, H, I, J being of the orders 2, 4, 6, in the roots, we may put

$$N = AI^3 + BJ^2 + CHIJ + DH^2I^2 + EH^3J \quad\text{................. (1),}$$

H^6 of the twelfth order in any root being omitted.

Taking the form $x^4 + 4x^3 + 6x^2 = 0$, for which $H = 0$, $I = 3$, $J = -1$, we have, if the roots are $0, 0, x_1, x_2$,

$$N = 4x_1^3x_2^3(x_1+x_2)^2(x_1-2x_2)^2(x_2-2x_1)^2 = 24^3 . 22^2 = 27A + B \ \text{... (2).}$$

So, from $x^4 - 6Hx^2 + 1 = 0$, we have, if the roots are $\pm x_1$, $\pm x_2$,

$$N = 2^4 x_1^2x_2^2[9(x_1^2+x_2^2)^2 - 100x_1^2x_2^2]^2 = 2^8(9^2H^2 - 25)^2,$$

whence, rejecting the factor 2^8 from this and (2), and equating coefficients with (1), $\quad A = 625, \quad 9A+B+D-E = -50 \cdot 9^2,$

$$27A-2B-C+6D-2E = 9^4, \quad 27A+B+C+9D+3E = 0.$$

From the last three $63A+16D = 81.31$. From (2), $B = 9261$; then $D = -2304$; so for the rest, giving the required result on restoring a.

9995. (C. L. Dodgson, M.A.)—A certain school contains not less than 90 boys nor more than 130. Latin, Greek, and French are taught, but no other languages. For every boy learning Latin, at least two learn Greek, but not French; for every three learning Greek, at least one learns French, but not Latin; and, for every two learning French, at least three learn Latin, but not Greek. Exactly half the school learn no languages. Find how many boys are learning each language.

Solution by J. C. St. Clair, L. Wiener, *and others.*

Let f, g, l represent the number of boys learning French, Greek, and Latin respectively. From the data it follows that

$$2l,\ g,\ 3f \text{ not} > \text{an unknown portion of } g,\ 3f,\ 2l;$$

adding, $\quad 2l+g+3f$ is not > a portion of $(2l+g+3f)$,

therefore the unknown portion is the whole, no boy learns two languages, and $g = 2l = 3f$. Now $N = 2(g+l+f) = 11f$, and is even. Therefore N is divisible by 22 and $= 110$; hence $f = 10$, $g = 30$, $l = 15$.

10026. (Professor Mannheim.)—On donne une sphère et un point fixe S. On coupe la sphère par un plan P, et l'on prend le petit cercle d'intersection, ainsi obtenu, comme directrice d'un cône qui a S pour sommet. Ce cône coupe de nouveau la sphère suivant un petit cercle dont le plan est Q. Démontrer que, si l'on fait varier le plan P, de façon qu'il passe par un point fixe, le plan Q passe aussi toujours par un même point.

Solution by Professor Genese, M.A.; F. R. J. Hervey; *and others.*

The planes P, Q meet in a straight line UV on the polar plane R of S. Also the planes SUV, P, R, Q are harmonic. Let F be the fixed point in P, and SF meet R in T and Q in X. Then $\{SXTF\}$ is harmonic. Therefore X is a fixed point.

7790. (R. Knowles, B.A.)—In Question 7593, if PQ meet the axis in a fixed point F, and a circle CDF cuts the parabola again in G, H; prove

(1) that the envelope of the chord GH is a parabola; (2) if F be the focus, the envelope becomes the original parabola.

Solution by G. G. STORR, M.A.; Rev. T. GALLIERS, M.A.; *and others.*

In the solution to Question 7593 (Vol. XLI., page 78) it is shown that the equation to CD is $ky+2ax+2a(2a-h)=0$. The equations to GH and the circle CDF will, if $\lambda = -(k^2+4a^2)^{-1}$, be

$$ky-2ax+l=0,\quad y^2-4ax+\lambda\left\{ky+2ax+2a(2a-h)\right\}(ky-2ax+l)=0 \quad (1),$$

The equation to PQ is $ky-2ax-2ah=0$; at F, $y=0$, therefore $x=-h$, a constant by the question. Since (1) passes through $(-h, 0)$ we have $l=h(k^2+4a^2)/(a-h)-2ah$, and the equation to GH becomes

$$ky-2ax+h(k^2+4a^2)/(a-h)-2ah=0.$$

The envelope of this line is the condition that this quadratic in k should have equal roots, or $(a-h)^2y^2+8ah(a-h)x-8ah^2(a+h)=0$, a parabola. If F be the focus, $-h=a$, and this equation becomes $y^2=4ax$, the original parabola.

10059. (R. W. D. CHRISTIE.) — Prove that every perfect number except the first two is the sum of an even number of odd cubes.

Solution by J. D. H. DICKSON, M.A.; Prof. RADAKRISHNAN; *and others.*

Let $2^{2n+1}-1$ be a prime number, then $2^{2n}(2^{2n+1}-1)$ is a perfect number. Also, the sum of the first 2^n odd cubes is

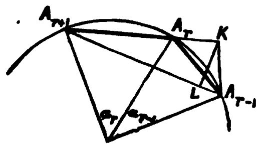
$$1^3+3^3+5^3+\ldots+(2\,.\,2^n-1)^3 = 2^{2n}(2\,.\,2^{2n}-1) = 2^{2n}(2^{2n+1}-1).$$

The first perfect number is 6, for which $n=\frac{1}{2}$. In this case the theorem fails, but it is true for all other perfect numbers, because, since $2^{2n+1}-1$ is a prime number, therefore $2n+1$ cannot be even, and therefore n has in every other case except that of 6 an integral value.

9913, 9970, 10001. (A. RUSSELL, M.A.)—If a polygon be inscribed in a circle of radius R, prove that (9913) $\Sigma(a_{r-1}^2-a_r^2)\cot A_r=0$; where a_{r-1}, a_r are two consecutive sides, and A_r the included angle; (9970) its area is $R^2\Sigma\dfrac{a_{r-1}a_r\sin A_r-\frac{1}{2}a_{r-1}^2\sin 2A_r}{a_{r-1}^2+a_r^2-2a_{r-1}a_r\cos A_r}$; (10001) if the polygon have an even number of sides, and θ have any value

$$R^2=\frac{\Sigma(-1)^r a_{r-1}a_r\sin(A_r+\theta)}{\Sigma(-1)^r\sin(2A_r+\theta)}.$$

Solution by J. D. Hamilton Dickson, M.A.



(9913) Let the side a_r subtend an angle α_r at the centre of the circum-circle; then $2A_r = 2\pi - \alpha_{r-1} - \alpha_r$;

hence $(a_{r-1}^2 - a_r^2)\cot A$

$= -4R^2[\sin^2\frac{1}{2}\alpha_{r-1} - \sin^2\frac{1}{2}\alpha_r]\cot\frac{1}{2}(\alpha_{r-1}+\alpha_r)$

$= -2R^2(\sin\alpha_{r-1} - \sin\alpha_r)$;

$$\therefore\ \Sigma(a_{r-1}^2 - a_r^2)\cot A_r = -2R^2.\Sigma(\sin\alpha_{r-1} - \sin\alpha_r) = 0.$$

[For Proposer's Solution, see Vol. LI., page 60.]

(9970) Join $A_{r-1}A_{r+1}$; drop perpendiculars, $A_{r-1}K$ on A_rA_{r+1}, and KL on $A_{r-1}A_{r+1}$. Then area of polygon $= \frac{1}{2}R^2.\Sigma\sin\alpha_{r-1}$.

Now, $\sin\alpha_{r-1} = \sin 2KA_{r+1}A_{r-1} = 2\sin KA_{r+1}A_{r-1}.\cos KA_{r+1}A_{r-1}$

$$= 2\frac{KL}{A_{r+1}K}.\frac{A_{r+1}K}{A_{r+1}A_{r-1}} = 2\frac{KL.A_{r+1}A_{r-1}}{A_{r+1}A_{r-1}^2} = 2\frac{A_{r-1}K.A_{r+1}K}{A_{r+1}A_{r-1}^2}$$

$$= 2\frac{a_{r-1}\sin A_r(a_r - a_{r-1}\cos A_r)}{a_{r-1}^2 + a_r^2 - 2a_{r-1}a_r\cos A_r} = 2\frac{a_{r-1}a_r\sin A_r - \frac{1}{2}a_{r-1}^2\sin 2A_r}{a_{r-1}^2 + a_r^2 - 2a_{r-1}a_r\cos A},$$

whence the result follows.

(10001.) This result is true if we prove these two equations

$$a_{r-1}a_r\sin A_r - a_ra_{r+1}\sin A_{r+1} + \ldots = R^2\{\sin 2A_r - \sin 2A_{r+1} + \ldots\}$$

and $a_{r-1}a_r\cos A_r - a_ra_{r+1}\cos A_{r+1} + \ldots = R^2\{\cos 2A_r - \cos 2A_{r+1} + \ldots\}$,

the number of terms on either side of both these equations being even.

Replacing $a_{r-1}a_r\sin A_r$, &c.,

by $4R^2.\sin\frac{1}{2}\alpha_{r-1}\sin\frac{1}{2}\alpha_r\sin\frac{1}{2}(\alpha_{r-1}+\alpha_r)$, &c.,

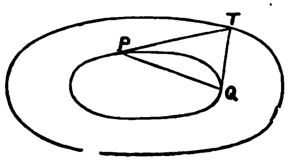
and $a_{r-1}a_r\cos A_r$, &c.,

by $-4R^2.\sin\frac{1}{2}\alpha_{r-1}\sin\frac{1}{2}\alpha_r\cos\frac{1}{2}(\alpha_{r+1}+\alpha^r)$, &c.,

the results follow at once.

10011. (G. G. Storr, M.A.)—From a point T on the ellipse $b^2x^2 + a^2y^2 = 4a^2b^2$, tangents TP, TQ are drawn to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$; prove that $\Delta TPQ = \frac{3}{4}\sqrt{3}.ab$.

Solution by W. J. Greenstreet, M.A.; R. Knowles, B.A.; *and others.*



By Smith's *Conics*, Problem 43, C. VI., if T be (h, k),

$$\Delta TPQ = (b^2h^2 + a^2k^2 - a^2b^2)^{\frac{3}{2}} \times\{1 - a^2b^2/(b^2h^2 + a^2k^2)\};$$

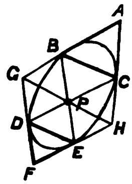
but, since T lies on the first curve,

$$b^2h^2 + a^2k^2 = 4a^2b^2,$$

therefore $\Delta TPQ = \frac{3}{4}\sqrt{3}\,ab$,

10003. (E. LEMOINE.)—Appelons, avec M. Neuberg, triangle semi-conjugué ou semi-autopolaire par rapport à une conique, le triangle aAa', où a et a' sont les intersections de la polaire de A avec cette conique ; on a le théorème : Deux triangles semi-conjugués par rapport à une conique sont inscriptibles à une autre conique et circonscriptibles à une troisième.

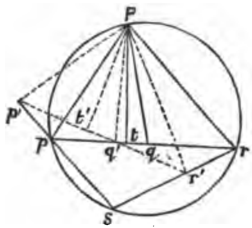
Solution by Professor SCHOUTE ; J. J. BARNEVILLE ; *and others.*



As the quadrilateral BCDE is inscribed in the given conic, the points P, G, H are collinear (see the diagram) ; for P is common to a pair of opposite sides, and G and H are the points common to tangents in opposite vertices. Therefore, by BRIANCHON'S theorem, the hexagon BCHEDGB is circumscribed to a new conic. And then the poles of the six sides, *i.e.*, the points A, B, C, D, E, F, lie on still another conic, the reciprocal polar of the former with reference to the given one.

10030. (Professor STEGGALL.)—Prove that, if from a fixed point on the circumscribing circle of a triangle, lines be drawn to cut the sides in order at the same angle, the points of intersection lie on a line which envelopes a parabola.

Solution by F. R. J. HERVEY ; Professor SCHOUTE ; *and others.*



From P on circumcircle of triangle prs draw Pq, making $\angle Pqp = Prs = Ppp'$, and Pt perpendicular to pr ; and suppose p, q, r to move along sp, rp, rs, respectively. If Pp, Pq, Pr turn in the same sense through the same angle, their lengths will change in the same ratio. Hence, p, q, r will remain collinear, the resulting *dotted* figure P$p't'q'r'$ being always similar to P$ptqr$. Hence, also, t will describe a straight line tt' (making with Pt the same angle as qp with Pq, &c.), which is therefore tangent at vertex to a parabola, with focus P, enveloped by $p'q'r'$.

10037. (Professor IGNACIO BEYENS.)—Si (P_1), (P_2), (P_3) sont les pieds des perpendiculaires abaissées du point de Lemoine d'un triangle sur les côtés BC, AC, AB, on aura la relation

$$(BP_1/a) + (CP_2/b) + (AP_3/c) = (CP_1/a) + (AP_2/b) + (BP_3/c) = \tfrac{3}{2}.$$

Solution by R. TUCKER, M.A.; R. KNOWLES, B.A.; *and others.*

$$BP_1/a = (c^2+ac\cos B)/k, \quad CP_1/a = (b^2+ab\cos C)/k,$$

(see "Triplicate Ratio" Circle, *Quarterly Journal of Mathematics*, Vol. XIX., No. 76, p. 345), therefore sinister side of Question

$$= (c^2+ac\cos B+a^2+ab\cos C+b^2+bc\cos A)/k$$
$$= (k+\tfrac{1}{2}k)/k = \tfrac{3}{2} = \text{dexter side.}$$

Also $$\left.\begin{array}{r} b\,.\,BP_1+c\,.\,CP_2+a\,.\,AP_3 \\ +c\,.\,CP_1+a\,.\,AP_2+b\,.\,BP_3 \end{array}\right\} = 3abc\,(a+b+c)/k;$$

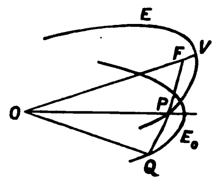
and $$a^2\,.\,AP_3^2+b^2\,.\,BP_1^2+c^2\,.\,CP_2^2 = a^2\,.\,AP_2^2+b^2\,.\,BP_3^2+c^2\,.\,CP_1^2.$$

Also $$AP_3\,.\,BP_1\,.\,CP_2+AP_2\,.\,BP_3\,.\,CP_1 = 3abc\tan^2\omega/4;$$

and $$BP_1\,.\,CP_1+CP_2\,.\,AP_2+AP_3\,.\,BP_3 = a^2b^2c^2\,(7-\cos A\cos B\cos C)/k^2,$$
$$[k \equiv a^2+b^2+c^2.]$$

10012. (W. J. GREENSTREET, M.A.)—Find the loci of the vertices and foci of concentric and similar ellipses passing through a fixed point.

Solution by Professors SCHOUTE, MATZ, *and others.*



In the affixed diagram, O is the common centre, P the given fixed point, E_0 the ellipse concentric and similar to the given series of which P is a focus, E an ellipse of the given series, F one of its foci, and Q the point of E_0 that corresponds to the point P of E. The similar triangles POQ and FOP give the relations $\angle POF = \angle QOP$, $OF\,.\,OQ = OP^2$. Therefore, and as OP is evidently an axis of symmetry for the locus of F, this locus is obtained in transforming E_0 by reciprocal radii vectores, O being the centre and OP^2 the power of the transformation; it is an unicursal bicircular quartic, the third double point of which is the conjugated point O. And as the ratio of OV to OF is constant, the locus of the vertex V is similar to that of F, O being the centre of similitude.

10013. (E. M. LANGLEY, M.A.)—Prove, geometrically, that the symmedian point of a triangle is the centroid of its projections on the sides.

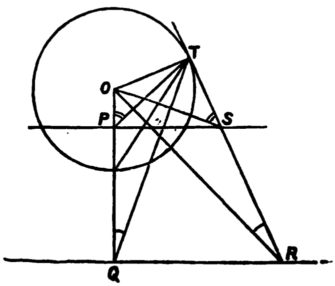
Solution by Professor SCHOUTE; J. J. BARNEVILLE; *and others.*

When K is the point of Lemoine, and K_a, K_b, K_c are its projections on the sides, KK_a, KK_b, KK_c are proportional to a, b, c (definition of K);

therefore the triangles K_bKK_c, K_cKK_a, K_aKK_b are equal, $bc \sin A$, $ca \sin B$, $ab \sin C$ being equal. This proves K to be the centroid of the triàngle $K_aK_bK_c$, etc.

10024. (J. Cirilli.)—Etant donnés un cercle et une droite, déterminer une seconde droite parallèle à la première de façon qu'une tangente quelconque au cercle coupe les deux droites en deux points dont le rapport des distances au centre du cercle soit constant.

Solution by J. C. St. Clair; G. E. Crawford, B.A.; *and others.*



The line through P, the pole of the given line, is the line required. Let O be the centre, and let OP meet the given line in Q. Let the tangent at any point T cut the given line in R, and the parallel in S. Since OQRT, OPST are concyclic, it is easily shown that PTQ, SOR are similar triangles,

$\therefore \quad TP : TQ = OS : OR.$

Again, since P, Q are inverse points, $OP : OT = OT : OQ$, therefore OPT, OTQ are similar triangles, and

$$TP : TQ = OP : OT = OT : OQ;$$

therefore $\qquad OS : OR = OT : OQ$, constant.

10014. (Capitaine de Rocquigny.)—On forme le tableau suivant :

1,
2, 3, 4,
3, 4, 5, 6, 7
... ...

. Démontrer que la somme des termes d'une horizontale est un carré impair.

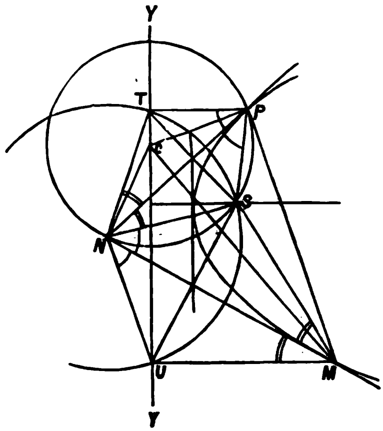
Solution by Thomas S. Fiske, A.M., Ph.D.; R. W. D. Christie; *and others.*

In the nth horizontal the first term is n, the number of terms is $2n-1$, the last term is $3n-2$. Consequently the sum of the terms will be $(2n-1)\frac{1}{2}(4n-2)$ or $(2n-1)^2$.

10016. (A. E. Jolliffe.)—O is a point on the directrix of a parabola and S the focus. A circle with centre O passes through S, and cuts the parabola in P and Q. The tangents at P to the circle and parabola meet

the parabola and circle respectively in M and N. Show by pure geometry that MN is a common tangent to both curves.

Solution by Professor Schoute; G. E. Crawford; *and others.*



Let, in the diagram, YY represent the directrix, and NM be considered as the second tangent to the parabola passing through N, M being its point of contact. Then it is to be proved that NM and PM touch the circle (O) in N and P. This can be derived from the generally known facts, that, PT and MU being perpendicular to the directrix, NS, NT, NU are equal, and this is also the case with the angles denoted in the figure by a single arc, and with those denoted by a double arc.

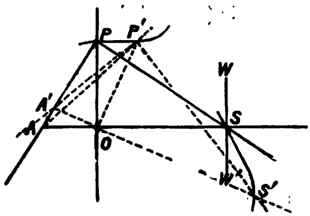
The equality of the angles MNS and NPS proves that the circle (O) is touched in N by NM.

The equality of the arcs PSN of (O) and TSU of (N), that is easily deduced from equality of angles, proves that the triangles PNO and TUN are similar; so the angles PNO and TUN are equal. But the circle described on OM as diameter passes through N and U; therefore the angles TUN and OMN are equal. And if the angles PNO and OMN are equal, PN is perpendicular to OM; therefore PM touches (O) in P.

[The two tangents of the parabola that pass through O, and the line at infinity, form an autopolar triangle of (O) circumscribed to the parabola. And the isotropic lines through S and the line at infinity form a triangle inscribed to (O) and circumscribed to the parabola. This proves the theorem (see Grunert's *Archiv*, Vol. iv., p. 318, § 40).]

9894. (Professor De Wachter.)—A system of two rectangular rods AOS and IOP moves about O as a pivot. The distance OA being constant, A is kept on the rod AP of a right angle APS whose vertex P ranges along OP. A disc WW′ (radius = OA) moves under OS with its axis parallel to it. If P traces out any curve PP′ in the plane, OP describes the area POP′. The disc in S, rolling on the plane, revolves in the meantime through a circular sector whose area = POP′ (Polar planimeter).

Solution by the PROPOSER; Professor SARKAR; *and others.*



Putting $OA = a$, $OS = r$, and $POP' = \omega$, the element of area described by OP is $= \frac{1}{2}\rho^2 . d\omega$. At the same time, the elementary sector rolled over by the wheel in S, is $\frac{1}{2}a . r\, d\omega$. Now, as the figure shows, $\rho^2 = ar$. Hence the proposition. The same property, by means of a different mechanical contrivance, has been applied by the Proposer to the construction of a polar integrator.

9981. (Professor GENESE, M.A.)—TP, TQ, T'P', T'Q' are tangents to a conic, (centre C, foci S, S'). Prove that T, P, Q, T', P', Q' will lie on a circle if (1) CT, CT' be on opposite sides of SS' and equally inclined to it, and (2) $CT . CT' = CS^2$.

Solution by R. H. W. WHAPHAM, B.A.; G. G. STORR, M.A.; *and others.*

Let the conic be $ax^2+by^2-1 = 0$; T, $x'y'$; T', $x''y''$; then PQ is

$$axx'+byy'-1 = 0; \quad P'Q' \text{ is } axx''+byy''-1 = 0;$$

therefore any conic through P, Q, P', Q' is

$$\lambda(ax^2+by^2-1)-(axx'+byy'-1)(axx''+byy''-1) = 0.$$

This will pass through T if $\lambda = ax'x''+by'y''-1$,

and the symmetry of this result shows that it will pass through T' also. The conic is a circle, if

$$x'y''+x''y' = 0, \text{ and } x'x''-y'y'' = a^{-1}-b^{-1} \quad\text{.........}\quad (1, 2).$$

From (1) and (2) we see that CT and CT' must be on opposite sides of SS' and equally inclined to it. Again, squaring and adding (1) and (2), we get $(x'^2+y'^2)(x''^2+y''^2) = (b^{-1}-a^{-1})^2$, or $CT . CT' = CS^2$.

9949. (H. W. SEGAR.)—PN is an ordinate of a parabola. NQ, NR are two lines drawn from N so that QNP, RNP are equal angles; show that $SQ . SR - AS . (SQ+SR)$ is constant for all values of the angles.

Solution by R. F. DAVIS, M.A.; R. KNOWLES, B.A.; *and others.*

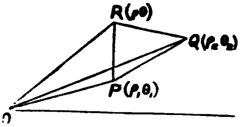
QN passes through a point R' symmetrically situated on the opposite side of the axis to R. Hence the product of the abscissæ of Q, R is $= AN^2$ = constant; or $(SQ-AS)(SR-AS)$ = constant, &c.

9744. (Professor Abinash Basu.)—Show that (1) the equation

$$\sin(\theta-\theta_1)\left\{\rho^2+\rho_2^2-2\rho\rho_2\cos(\theta-\theta_2)\right\}^{\frac{1}{2}} + \sin(\theta_2-\theta)\left\{\rho^2+\rho_1^2-2\rho\rho_1\cos(\theta_1-\theta)\right\}^{\frac{1}{2}} = 0$$

represents a rectangular hyperbola; and find (2) whether the locus represents anything more.

Solution by Professors Ramaswami Aiyar *and* Nilkantha Sarkar.



Let O be the pole; P, Q the points (ρ_1, θ_1), (ρ_2, θ_2), and R the point (ρ, θ); then

$$\text{RQ}\sin\text{ROP}-\text{RP}\sin\text{ROQ}=0,$$

therefore $\dfrac{\sin\text{RPO}}{\text{RO}}=\dfrac{\sin\text{RQO}}{\text{RO}}$;

therefore the angles RPO, RQO are either equal or supplementary. If they are equal when they are on opposite sides of OR, and supplementary when on the same side, the locus of R is a rectangular hyperbola, having PQ for a diameter and OR for any chord; but, if not, R is obviously on the circle about OPQ.

9938. (Professor Matz.)—The two points of suspension, supposed in the same horizontal line, are lowered over a horizontal table, until a length z of the chain, the whole length of which is l, is in contact with the table; prove that, if b be the height above the plane of the points of suspension, the horizontal tension is equal to the weight of a length

$$\frac{1}{8}\frac{(l-z)^2}{b}-\frac{b}{2} \text{ of the chain.}$$

Solution by R. F. Davis, M.A.; Professor Mukhopadhyay; *and others.*

Let the horizontal tension be equal to the weight of a length c of the chain, and ϕ be the angle which the direction of the string at either point of support makes with the horizontal. Then, by the known properties of the catenary, arc of each curved portion $=\frac{1}{2}(l-z)=c\tan\phi$,

height of point of support above directrix $=b+c=c\sec\phi$.

Eliminating ϕ, we find c as required.

9670. (Professor Hudson, M.A.)—Prove that (1) the law of force under which the pedal of $p=f(r)$ can be described is

$$h^2\left\{\frac{2r^2}{p^5}-\frac{r}{p^4}\frac{dr}{dp}\right\};$$

and (2) if $p \propto r^n$, the law of force under which the pedal can be described varies inversely as (distance)$^{5-2/n}$.

Solution by Rev. T. Galliers, M.A.; G. G. Storr, M.A.; *and others.*

(1) Let P, P′ be the central forces under which $p = f(r)$ and its pedal can be described respectively; (p, r), (p', r') corresponding points on the curve and its pedal; then, from Differential Calculus and Dynamics, we have $p' = p^2/r$ and $P = h^2 p^{-3}\, dp/dr$, therefore

$$P' = h^2 p'^{-3}\, dp'/dr' = h^2 \left\{2r^2 p^{-5} - rp^{-4}\, dp/dr\right\},$$

as will be found on substitution.

(2) If $p = \mu^n r^n$, we shall find

$$P' = h^2 (2\mu^{-1} - n^{-1}\mu^{-1})\, r'^{-(5-2/n)}, \quad \text{or} \quad P' \propto r'^{-(5-2/n)}.$$

9726. (B. Reynolds, M.A.)—Prove that the coefficient of $\cos^{r-1}\alpha$ in the expansion of $\cos(n-1)\alpha$ is the arithmetic mean of the coefficients of $\cos^r\alpha$ in the expansions of $\cos n\alpha$ and $\cos(n-2)\alpha$. The same law holds in the expansions of $\dfrac{\sin(n-2)\alpha}{\sin\alpha}$, $\dfrac{\sin(n-1)\alpha}{\sin\alpha}$, and $\dfrac{\sin n\alpha}{\sin\alpha}$, in terms of $\cos\alpha$. Hence show a plan for rapidly writing out a complete set of expansions.

Solution by the Proposer.

The law itself may be verified by ordinary methods.

By varying the statement of the law, we get

$$\text{coefficient of } \cos^r\alpha \text{ in } \cos n\alpha = 2 \times \text{coefficient of } \cos^{r-1}\alpha \text{ in } \cos(n-1)\alpha$$
$$- \text{coefficient of } \cos^r\alpha \text{ in } \cos(n-2)\alpha,$$

which is in substance the discovery made public in the letter by Mr. G. A. Dierke, in the *Educational Times* for April, 1888, page 171. I append a sort of scheme for utilizing the law in writing out a complete set of expansions.

The arrows show the law of connexion of coefficients; thus,

$$18 = 2 \times 5 - (-8); \quad -48 = 2 \times (-20) - 8,$$

and so on. Let c stand for $\cos\alpha$; then we have

$$\begin{array}{llllllll}
\cos(0)\alpha & = 1, & & & & & & \\
\cos(1)\alpha & = 0 & +\,c, & & & & & \\
\cos 2\alpha & = -1 & & +\,2c^2, & & & & \\
\cos 3\alpha & = 0 & -\,3c, & & +\,4c^3, & & & \\
\cos 4\alpha & = 1 & & -\,8c^2 & & +\,8c^4, & & \\
\cos 5\alpha & = 0 & +\,5c & & -\,20c^3 & & +\,16c^5, & \\
\cos 6\alpha & = -1 & & +\,18c^2 & & -\,48c^4 & & +\,32c^6.
\end{array}$$

For second part of question, the law of connexion is precisely the

same as before; thus, $-12 = 2\times(-4)-4$; or, otherwise put, -12 is the third term of the A.P., of which $+4$ and -4 are the first two terms:

$$\begin{array}{lllllll}
\dfrac{\sin(0)\alpha}{\sin\alpha} & = 0, & & & & & \\
\dfrac{\sin(1)\alpha}{\sin\alpha} & = & +1, & & & & \\
\dfrac{\sin 2\alpha}{\sin\alpha} & = 0 & & +2c, & & & \\
\dfrac{\sin 3\alpha}{\sin\alpha} & = & -1 & & +4c^2, & & \\
\dfrac{\sin 4\alpha}{\sin\alpha} & = 0 & & -4c & & +8c^3, & \\
\dfrac{\sin 5\alpha}{\sin\alpha} & = & 1 & & -12c^2 & & +16c^4.
\end{array}$$

9972. (A. E. JOLLIFFE.) — If two quadrilaterals have a common diagonal, and are circumscribed to the same conic, prove (1) that the remaining eight vertices which do not lie on this diagonal lie on a conic; and hence (2) deduce the locus of the foci of all conics inscribed in a parallelogram.

Solution by Professors WOLSTENHOLME, GENESE, *and others.*

Let $x \pm y \pm z = 0$ be the sides of one of the circumscribed quadrilaterals; $px^2+qy^2+rz^2=0$ the conic; $(qr+rp+pq=0)$; $z=0$ the common diagonal of the two quadrilaterals; $(x_1, y_1, 0)$, $(x_2, y_2, 0)$ the ends of that diagonal of the second quadrilateral which lies along z. The tangents from these points to the conic will be

$$U_1 \equiv (px^2+qy^2+rz^2)(px_1^2+qy_1^2)-(pxx_1+qyy_1)^2 = 0,$$
$$U_2 = (px^2+qy^2+rz^2)(px_2^2+qy_2^2)-(pxx_2+qyy_2)^2 = 0;$$

and the equation of a conic through the ends of the other two diagonals will be $\lambda U_1 \mp U_2$, which, if the theorem be true, must be made to coincide with
$$\mu\left\{(x+y)^2-z^2\right\} = \left\{(x-y)^2-z^2\right\}.$$

This gives the equations $\dfrac{pq(\lambda y_1^2-y_2^2)}{\mu-1} = \dfrac{pq(\lambda x_1^2-x_2^2)}{\mu-1}$

$$= \frac{-r\left\{\lambda(px_1^2+qy_1^2)-(px_2^2+qy_2^2)\right\}}{\mu-1} = \frac{-pq(\lambda x_1y_1-x_2y_2)}{\mu+1},$$

which are obviously satisfied by giving proper values to λ and μ, remembering that $qr+rp+pq=0$, for all values of (x_1y_1), (x_2y_2). These values are $\lambda = (x_2^2-y_2^2)/(x_1^2-y_1^2)$, $\mu = (x_1-y_1)(x_2-y_2)/(x_1+y_1)(x_2+y_2)$.

We may take two such quadrilaterals to be (1) a real parallelogram, (2) the four tangents drawn from the foci, the common diagonal being the straight line at infinity. Hence we see that the four foci of any conic

and the angular points of any circumscribed parallelogram all lie on one conic. This conic will be a rectangular hyperbola, since the two unreal foci are antipoints to the two real, and any conic through the four foci is a rectangular hyperbola. Thus the locus of the foci (real and impossible) of any conic inscribed in a given parallelogram is the rectangular hyperbola which circumscribes the parallelogram.

[It would be much easier to prove this last theorem independently, and deduce the former from it. If F be a focus of any inscribed conic, the product of the perpendiculars from F on the two pairs of parallel sides must be equal, which gives the locus at once as stated.

The theorem may be otherwise proved as follows:—

(1). Let AB, A′B′ be the vertices of the quadrilateral, on the common diagonal, and let I, J be the point-pair for which $\{\dot{A}\dot{B}IJ\}$ and $\{\dot{A}'\dot{B}'IJ\}$ are harmonic ranges. Now project the figure so that I, J become the circular points at infinity. Then the quadrilaterals project into rectangles, and their vertices lie on the orthocycle. Therefore, &c.

(2) Again, the orthocycle passes through I, J; therefore the eight-point conic (1) divides AB harmonically. Now, let A, B be the circular points at infinity; then the eight-point conic becomes a rectangular hyperbola, and we deduce that the foci of a conic inscribed in a parallelogram lie on the rectangular hyperbola circumscribing it.]

9895. (Professor GOB.)—Si l'on prend les côtés BC, CA, AB d'un triangle ABC pour bases de trois séries de triangles ayant même angle de Brocard V que ABC, les sommets de ces triangles se trouvent, comme on le sait, sur trois circonférences déterminées, appelées circonférences de Neuberg. Soient N_a, N_b, N_c les centres de ces cercles, O et R le centre et le rayon du cercle ABC. Cela posé:

(1) $ON_a : ON_b : ON_c = a^3 : b^3 : c^3$; (2) $a^2 . ON_bN_c = b^2 . ON_cN_a = c^2 . ON_cN_a$;

(3) $\dfrac{ON_a}{a} + \dfrac{ON_b}{b} + \dfrac{ON_c}{c} = \cot V$; (4) $ON_a . ON_b . ON_c = R^3$;

(5) les tangentes menées en A, B, C, aux cercles N_a, N_b, N_c, passent par le point de Steiner.

Solution by R. F. DAVIS, M.A.

In Vol. 47, Appendix II., p. 136, I have shown that Neuberg's circles are concentric with the circular segments described (inwards) upon the sides each containing the Brocard angle V.

Hence ON_a is the perpendicular bisector of BC, and $= \frac{1}{2}a(\cot V - \cot A)$ $= \frac{1}{2}a(\cot B + \cot C) = R(a^2/bc)$.

Hence (1) $ON_a : ON_b : ON_c = a^3 : b^3 : c^3$.

(2) $a^2 . ON_bN_c = a^2 . \frac{1}{2} . R(b^2/ca) . R(c^2/ab) \sin A = R\Delta$.

(3) $ON_a/a + \ldots = R(a^2 + b^2 + c^2)/abc = \cot V$. (4) is obvious.

And (5) since the perpendiculars from N_a on CA, AB are as the sines of $(\frac{1}{2}\pi - V) \sim C$, $(\frac{1}{2}\pi - V) \sim B$, therefore AN_a passes through the Tarry point

whose trilinear coordinates are inversely as cos $(A+V)$, ... ; and the perpendicular through A to AN_a passes through the Steiner point (MILNE's *Companion*, p. 177).

9869. (A. RUSSELL, B.A.)—Prove that

$$L_{x=0}\left\{a\sin 2x\cos(3x-a\sin 2x)+\sin(x-a\sin 2x)\right.$$
$$\left.-\sin x\cos(2x-a\sin 2x)\right\}/\sin^3 x$$
$$=L_{x=0}\left\{-a\sinh 2x\cosh(3x-a\sinh 2x)-\sinh(x-a\sinh 2x)\right.$$
$$\left.+\sinh x\cosh(2x-a\sinh 2x)\right\}/\sinh^3 x$$
$$=2\left\{\tfrac{4}{3}a^3+2(1-a)^3-1\right\}.$$

Solution by the PROPOSER.

Let

$$u=a\sin 2x\cos(3x-a\sin 2x)+\sin(x-a\sin 2x)-\sin x\cos(2x-a\sin 2x)$$

$= 0$ when $x=0$, then we have

$$\frac{du}{dx}=2a\cos 2x\cos(3x-a\sin 2x)-2a\sin 2x\sin(3x-a\sin 2x)(3-2a\cos 2x)$$
$$+\cos(x-a\sin 2x)(1-2a\cos 2x)-\cos x\cos(2x-a\sin 2x)$$
$$+\sin x\sin(2x-a\sin 2x)(2-2a\cos 2x)=0 \text{ when } x=0;$$

$$\frac{d^2u}{dx^2}=-4a\sin 2x\cos(3x-a\sin 2x)-2a\cos 2x\sin(3x-a\sin 2x)(3-2a\cos 2x)$$
$$-2a\cos 2x\sin(3x-a\sin 2x)(3-2a\cos 2x)$$
$$-a\sin 2x\cos(3x-a\sin 2x)(3-2a\cos 2x)^2$$
$$-4a^2\sin^2 2x\sin(3x-a\sin 2x)+4a\sin 2x\cos(x-a\sin 2x)$$
$$-(1-2a\cos 2x)^2\sin(x-a\sin 2x)+\sin x\cos(2x-a\sin 2x)$$
$$+2\cos x\sin(2x-a\sin 2x)(2-2a\cos 2x)$$
$$+\sin x\cos(2x-a\sin 2x)(2-2a\cos 2x)^2$$
$$+4a\sin 2x\sin x\sin(2x-a\sin 2x)=0 \text{ when } x=0;$$

and

$$\left(\frac{d^3u}{dx^3}\right)_{x=0}=-8a-6a(3-2a)^2+8a-(1-2a)^3+1+3(2-2a)^2$$
$$=12-72a+72a^2-16a^3.$$

Also $\left[\dfrac{d^3}{dx^3}(\sin^3 x)\right]_{x=0}=6$; thus the limit of given fraction

$$=\tfrac{1}{6}(12-72a+72a^2-16a^3)=2\left\{\tfrac{4}{3}a^3+2(1-a)^3-1\right\}.$$

In an exactly similar manner we can prove that limit of other fraction is equal to the same thing.

9837. (Professor WOLSTENHOLME.)—Given the circumcircle, and the centroid, of a triangle; prove that the locus of the centres of the four circles which touch the sides is a certain Cartesian oval, whose triple

focus is the given centre of the circumcircle, and whose single foci are (1) the centre of the nine points' circle (which circle is also given), and (2), (3) the two points which are coaxial with the circumcircle and the circle of which the circumcentre and orthocentre are ends of a diameter. [This locus is therefore not the general Cartesian, the distances a, b, c of the three single foci (1), (2), (3) from the triple focus satisfying the equation $a^{-1} = b^{-1}+c^{-1}$. This condition makes the curvatures at the vertices equal, two and two.]

Solution by the PROPOSER.

Let O be the centre of the circumcircle, R its radius, A the centre of the nine points' circle (so that the given centroid divides OA in the ratio 2 : 1), B, C the two point circles coaxial with the circumcircle and the circle of which O and the orthocentre are ends of a diameter, and let a, b, c denote the lengths OA, OB, OC. Then $bc = R^2$, and $a(b+c) = R^2$. Now, let P be the centre of a circle which touches the sides of the triangle, then $OP^2 = R^2 \pm 2Rr$, $AP = \frac{1}{2}R \pm r$, where r is the radius of the circle, and the ambiguities are alike. Thus $OP^2 = 2R \, . \, AP$, and if we express this in rectangular coordinates, O being origin, and A on the axis of x, we get the equation

$$(x^2+y^2)^2 = 4R^2\left[(x-a)^2+y^2\right] = 4bc\left[(x-a)^2+y^2\right],$$

which coincides with the general equation of a Cartesian,

$$(x^2+y^2-bc-ca-ab)^2+4abc\,(2x-a-b-c) = 0,$$

under the single condition $a(b+c) = bc$. Thus the locus is as stated. When the foci B, C are real, the triangle must be acute-angled, its angles θ, ϕ, ψ satisfying the equation

$$8\cos\theta\cos\phi\cos\psi = 1-4a^2/bc \equiv \left\{(b-c)/(b+c)\right\}^2.$$

The vertices of the Cartesian are at the distances $\pm(bc)^{\frac{1}{2}} \pm (ca)^{\frac{1}{2}} \pm (ab)^{\frac{1}{2}}$ from O, the number of negative signs being 1 or 3. The focus A is nearest to O, and, if $c > b$, the foci A, B lie within the inner loop of the oval (which is the locus of the incentre only), the focus C is external to both loops, and from it can be drawn tangents to both loops, touching them in points of maximum or minimum curvature; the radius of curvature being $\sqrt{a}\left[\sqrt{c}+\sqrt{c-b}\right]$ on the outer loop and a minimum, and being $\sqrt{a}\left[\sqrt{c}-\sqrt{c-b}\right]$ on the inner loop and a maximum. The sides of the triangle always touch an ellipse whose auxiliary circle is the fixed nine points' circle and of which one focus is at O. This ellipse has, I believe, a definite label in the new *Surname Geometry*.

9942. (Professor DE LONGCHAMPS.)—Sommer la série convergente $(\alpha n^2+\beta n+\gamma)/n!$ quand on suppose $2\alpha+\beta+\gamma = 0$.

Solution by the PROPOSER.

Posant $\theta_n = 1/n!$ on observe que $u_n = \alpha n\theta_{n-1}+\beta\theta_{n-1}+\gamma\theta_n$; ou,

$$u_n = \alpha n\theta_{n-1}+\beta\theta_{n-1}-(2\alpha+\beta)\theta_n.$$

Cette égalité donne $\quad u_n = \alpha(n-1+1)\,\theta_{n-1} - 2\alpha\theta_n + \beta(\theta_{n-1} - \theta_n),$

ou $\quad u_n = \alpha(\theta_{n-2} + \theta_{n-1} - 2\theta_n) + \beta(\theta_{n-1} - \theta_n).$

En changeant, dans cette formule, successivement, n en $n-1$, $n-2$, ...; puis, faisant la somme des égalités ainsi obtenues, on a la somme des n premiers termes de la série proposée.

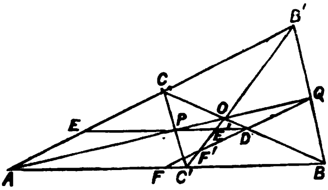
Si, nous même, ne faisons pas erreur, on a $S_n = \beta + (n\alpha + \gamma)/n!$ et, pour $n = 8$, $S_\infty = \beta$.

10019. (Prof. Syamadas Mukhopâdhyây, B.A.)—ACB′, AC′B are two lines such that AC = AC′, AB = AB′; BC, B′C′ intersect at O; AO meets CC′, BB′ at P, Q; D is the mid-point of BC; DP intersects AC, B′C′ at E, E′; QD intersects AB, B′C′ at F, F′. Prove (1) that

$$DE' : DP = DP : DE, \text{ and } DF' : QD = QD : DF;$$

and hence (2) that B′C′ is the inverse of the "nine-point circle" of ABC, D being the centre and $\frac{1}{2}$(AB − AC) the radius of inversion.

Solution by R. H. W. Whapham, B.A.; *and* E. M. Langley, M.A.



$DE' : DP = BC' : AB = DP : DE,$

$DF' : DQ = B'C : AC = DQ : DF;$

$$DQ = B'C/2 = BC'/2 = DP = \tfrac{1}{2}(AB - AC);$$

therefore $DE' \cdot DE = DF' \cdot DF = DP^2 = \{(AB - AC)/2\}^2.$

The nine-point circle of ABC is the circle through E, F, D; therefore the inverse of the nine-point circle of ABC with respect to D as centre of inversion, and $\frac{1}{2}$(AB − AC) as radius, is the line B′E′F′C′.

9868. (C. E. McVicker, B.A.)—Two conics A, B are so situated that their four common tangents all touch the same circle C; prove that (1) to any conic confocal with A corresponds a conic having double contact with it and confocal with B, the centre of C being the pole of the chord of contact; (2) note the cases when (*a*) either conic reduces to a point pair; and when (*b*) C reduces to a point circle, *i.e.*, when A, B have double contact; and (3) prove that the converse proposition is also true.

Solution by the Proposer.

Let $\Sigma = 0$, $\Sigma' = 0$ be the tangential equations of the two conics, $\alpha = 0$ the centre of the circle, $\Omega = 0$ the circular points at infinity. The truth of the theorem appears from an inspection of the following equivalent identities: $\quad \Sigma + \Sigma' \equiv \alpha^2 + k\Omega, \quad (\Sigma + \lambda\Omega) + \{\Sigma' - (k+\lambda)\,\Omega\} \equiv \alpha^2.$

2419. (The late T. COTTERILL, M.A.)—1. If AA′, BB′, CC′ are the opposite intersections of a complete quadrilateral, prove that an infinite number of cubics can be drawn through these points and another point D, touching DA, DA′ at A and A′, and that amongst these cubics, there are two triads of straight lines and four cubics having respectively a point of inflexion at B, B′, C, C′.

2. Prove that the locus of the intersection of tangents at B, B′ is the conic DAA′BB′, and that of tangents at C, C′ is the conic DAA′CC′; also give the reciprocal results when the class cubic degenerates.

Solution by Professor SEBASTIAN SIRCOM.

1. Let $\alpha, \beta, \gamma, \delta$ be the lines of the quadrilateral; AA′, BB′, CC′ the intersections of α, β; γ, δ : β, γ; α, δ : α, γ; β, δ respectively, then the equation of the cubic may be written $(c\gamma + d\delta)\,\alpha\beta = k\,(a\alpha + b\beta)\,\gamma\delta$, where $a\alpha + b\beta = 0$ and $c\gamma + d\delta = 0$ are the tangents at A, A′; and the curve passes through their intersection D for all values of k.

The values $k = 0$, $1/k = 0$ give the triads of lines, and if $\delta \equiv l\alpha + m\beta + n\gamma$, the tangent at C will be $d\alpha = kb\gamma$, which will be an inflexional tangent if $kma = c$, so for C′, B, B′.

2. The tangent at C′ will be $c\beta = ka\delta$, whose intersection with that at C is on the conic $ad\alpha\beta = bc\beta\gamma$ passing through DAA′CC′; so for the tangents at B, B′.

Amongst the class cubics touching the lines joining the points x, y, z, w there will be two triads of points and four cuspidal class cubics, the lines xz, &c. being tangents at the cusps.

9632. (Professor NILKANTHA SARKAR, M.A.)—If $\alpha, \beta, \gamma, \delta$ be the tangents of the angles which the normals from any point to an ellipse make with the major axis, find an invariable relation between them.

Solution by Rev. T. GALLIERS, M.A.; G. G. STORR, M.A.; *and others.*

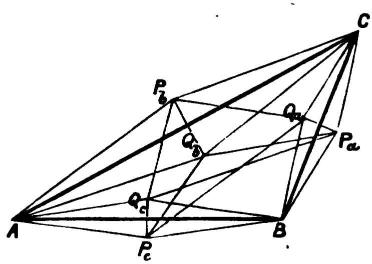
The ellipse being $b^2x^2 + a^2y^2 = a^2b^2$, it may be shown that $\alpha, \beta, \gamma, \delta$ are the roots of $b^2h^2x^4 - 2b^2hkx^3 + \{b^2k^2 + a^2h^2 - (a^2 - b^2)^2\}\,x^2 - 2a^2hkx + a^2k^2 = 0$, where (h, k) is the point through which the normals are drawn; hence the relation required is $(\alpha + \beta + \gamma + \delta)^2 a^2 = 4b^2(\alpha\beta\gamma\delta)$.

10035. (Professor GOB.)—Sur les côtés de ABC, on construit six triangles isoscèles semblables; soient P_a, P_b, P_c les sommets des triangles extérieurs à ABC, et soient Q_a, Q_b, Q_c les sommets des triangles intérieurs. Montrer que les quadrilatères $AP_bQ_aP_c$, $AQ_bP_aQ_c$ sont des parallélogrammes.

10096. (W. S. M'Cay, M.A.)—Let A′, B′, C′ be three corresponding points on the perpendicular bisectors of the sides of a triangle ABC, whose parameter is θ (*i.e.*, vertices of three isosceles triangles on the sides, with base angle θ). Prove that the lines joining ABC to the middle points of corresponding sides of A′B′C′ concur at a point on Kiepert's hyperbola whose parameter is $-\theta$.

Solution by Professors Schoute, Nash, *and others*.

(10035.) This is a particular case of the following due to Prof. H. van Aubel:—If on the sides BC, CA, AB of a given triangle are constructed outward the triangles BP_aC, CP_bA, AP_cB, and inward the triangles CQ_aB, AQ_bC, BQ_cA, so that these six triangles are similar to one another in the indicated order of the vertices, the figures $AQ_bP_aQ_c$, $AP_bQ_aP_c$, etc. are parallelograms (compare, in *Mathesis* I., p. 166, the elementary demonstration by Interdonato, and in the *Wiskundige Opgaven*, III., Question 36, the solution by means of equipollences by Mantel).



(10096.) This is an immediate consequence of the preceding, and of the generally known mode of generation of the hyperbola of Kiepert (compare *Mathesis* VII., p. 208 and *Schlömilch's Zeitschrift*, t. XXXII., p. 61).

On the Recent Geometry of the Triangle, embodying Solutions of Questions 9950, 10007, 10045, 10101, 10139.

By R. Tucker, M.A.

[*** The Questions will be found in the Contents to this Volume.]

(9950.) The equation to the join of the Lemoine point (K) and the orthocentre (O) is $\quad \alpha\cos^2 A\sin(B-C)+\ldots+\ldots=0,$

and Σ (9875) is $\quad \alpha\cos A/a\cos(B-C)=\ldots=\ldots,$

whence (1) at once follows.

The join of

D (0, cos C, cos B) and σ_1 $\left[a,\ b\cos(C-A)/\cos B,\ \ c\cos(A-B)/\cos C\right]$

is $\quad \alpha(b^2-c^2)\cos A-\beta a^2\cos B+\gamma a^2\cos C=0;$

hence $E\sigma_2$ is $\quad \alpha b^2\cos A+\beta(c^2-a^2)\cos B-\gamma b^2\cos C=0,$

and F_3 is $\quad -\alpha c^2\cos A+\beta c^2\cos B+\gamma(a^2-b^2)\cos C=0;$

these cointersect in $\alpha\cos A/a^2=\ldots=\ldots$ (the point π' say) ....(2).

From the above equations we see that the lines pass through $(-a, b, c)$, $(a, -b, c)$, $(a, b, -c)$, which are the associated points (P, Q, R, *Simmons*, p. 113) of K.

The join of Π and O is $\alpha(b^2-c^2)\cos A + \ldots + \ldots = 0$, which clearly passes through the circumcentre and the centroid. The join of Π and Σ will be found to be $bc\cos A\cos 2A\sin(B-C)\,\alpha + \ldots + \ldots = 0$, and it evidently passes through $(a\sec 2A, \ldots, \ldots)$ ....(3).

The nine-point centre of ABC (N), *i.e.*, the circumcentre of DEF, is $\cos(B-C), \ldots, \ldots$; hence N$\Sigma$, the circum. B. axis of DEF, is

$$\alpha\cos A\tan(B-C) + \ldots + \ldots = 0,$$

and the corresponding line for ABC is $\alpha\sin(B-C) + \ldots + \ldots = 0$; these intersect in β, the point $\sin 2A\cos(B-C), \ldots,$ ....(4).

The equations to $D\sigma_2$, $D\sigma_3$ are

$$\alpha\cos^2 A + \beta\cos B\cos(B-C) - \gamma\cos C\cos(B-C) = 0,$$

$$-\alpha\cos^2 A + \beta\cos B\cos(B-C) - \gamma\cos C\cos(B-C) = 0, \text{ hence } (5).$$

The simplest way to obtain (6) is perhaps by the following transformation. Let (α, β, γ), $(\alpha', \beta', \gamma')$ be the coordinates of a point P referred to ABC, DEF respectively, then

$$\beta'+\gamma' = 2\alpha\cos A,\ \gamma'+\alpha' = 2\beta\cos B,\ \alpha'+\beta' = 2\gamma\cos C \ldots\ldots\ldots(A),$$

and $\alpha' = -\alpha\cos A + \beta\cos B + \gamma\cos C$, &c......................(A′).

From (A) $\alpha\cos A \mid \sin^2 2A\sin 2B + \sin^2 2B\sin 2C$.

["T. R." circle xxiii., by | here, and elsewhere, I indicate that a constant factor is omitted for brevity.]

Hence, for the positive B. point, we have

$$\alpha\cos A/\sin 2B\,(\sin^2 2A + \sin 2B\sin 2C) = \ldots = \ldots,$$

and for the negative B. point,

$$\alpha\cos A/\sin 2C\,(\sin^2 2A + \sin 2B\sin 2C) = \ldots = \ldots.$$

This result was originally obtained by a tedious process. The coordinates of Σ (9875) come at once by this transformation.

By (A) the coordinates of G′ are $a^2\cos(B-C), \ldots, \ldots$; hence ΠG′ is given by $bc\cos A\sin(B-C)\,\alpha + \ldots + \ldots = 0$, and (7) follows.

(10007.) We have, by the data, $BD = a\tan\omega\cot C = m\cot C$, $DD' = m\cot A$, $D'C = m\cot B$, $\therefore BD' = 2R\tan\omega\, b/c$, and $BF = 2R\tan\omega\, b/a$, $\therefore BD'/BF = a/c$, and D′F is parallel to AC, and so on.

The equations to D′F, DE′ are $\beta = m\cot B\sin C$, $\gamma = m\cot C\sin B$; hence the three lines meet in $a^2\alpha\sec A = \ldots = \ldots$ (which is the point π in the "S. point axis" paper) ....(1).

$$\begin{aligned} \text{The } \triangle DEF &= \triangle PDE + \triangle PEF + \triangle PFD \\ &= ab\tan^2\omega\cot A\cot B\sin C/2 + \ldots + \ldots = \Delta\tan^2\omega \\ &= \triangle D'E'F' \ldots\ldots\ldots\ldots(2). \end{aligned}$$

We have $\sin(B+\phi_3)/\sin\phi_3 = BF/BD = bc/a^2\cos C$,

whence $\cot\phi_3/\cot\omega = \cot A/\cot C$.

Similarly, $\sin(A+\phi_3')/\sin\phi_3' = ac/b^2\cos C$,

and therefore $\cot\phi_3'/\cot\omega = \cot B/\cot C$,

$$\frac{\cot\phi_2}{\cot\omega} = \frac{\cot C}{\cot B},\ \frac{\cot\phi_1}{\cot\omega} = \frac{\cot B}{\cot A};\ \frac{\cot\phi_2'}{\cot\omega} = \frac{\cot A}{\cot B},\ \frac{\cot\phi_1'}{\cot\omega} = \frac{\cot C}{\cot A};$$

whence $\cot\phi_1 . \cot\phi_2 . \cot\phi_3 = \cot^3\omega = \cot\phi_1' . \cot\phi_2' \cot\phi_3'$,

$$\cot\phi_1 . \cot\phi_2' = \cot^2\omega = \cot\phi_2 \cot\phi_3' = \cot\phi_3 . \cot\phi_1',$$

$$\frac{\cot\phi_1 + \cot\phi_1'}{\cot\omega} . (\ ) . (\ ) = \frac{\cot\omega - \cot A \cot B \cot C}{\cot A \cot B \cot C} \text{ ....(3)},$$

and other like results.

$$D\pi/E'\pi = BF'\pi AF = \cot A/\cot B,$$

$$F\pi/D'\pi = \cot C/\cot A,\ E\pi/F'\pi = \cot B/\cot C, \text{ whence (4).}$$

Also we have $\dfrac{D\pi . E'\pi}{c^2} + \dfrac{F\pi . D'\pi}{b^2} + \dfrac{E\pi . F'\pi}{a^2} = \tan^2\omega$,

$$DE'/c + EF'/a + FD'/b = 2,$$

and each of the results in (4)

$$= abc \tan^3\omega \cot A \cot B \cot C = 8R^3 \tan^3\omega \cos A \cos B \cos C.$$

The join of D $(0, c^3, ab^2 \cos C)$ and E $(bc^2 \cos A, 0, a^3)$ is

$$\alpha a^3 c + \beta ab^3 \cos C \cos A - \gamma bc^3 \cos A = 0,$$

and of D′ $(0, ac^2 \cos B, b^3)$, F′ $(b^2 c \cos A, a^3, 0)$ is

$$\alpha a^3 b - \beta b^3 c \cos A + \gamma ac^3 \cos A \cos B = 0,$$

and these lines intersect on $\beta b^3 = \gamma c^3$, and therefore

$$Ap,\ Bq,\ Cr \text{ cointersect in } \alpha a^3 = \beta b^3 = \gamma c^3 \ldots (\pi_1) \text{...............(5).}$$

The join of EF is $-\alpha a^3 c \cos B + \beta ab^3 + \gamma bc^3 \cos A \cos B = 0$,

and of E′F′ is $\alpha a^3 b \cos C - \beta b^3 c \cos A \cos C - \gamma ac^3 = 0$,

which intersect on $b \tan^2 B\beta = c \tan^2 C\gamma$,

and therefore Ap_1, Bq_1, Cr_1 cointersect in $a^3\alpha \sec^2 A = \ldots = \ldots (\pi_2)$....(6).
The join of $\pi_1\pi_2$ is $\alpha a^3 (b^2 - c^2) + \ldots + \ldots = 0$, which passes through the centroid.

By the "S. point axis" paper, $\pi G = 2KG$, and $HG = 2OG$, therefore

G is the centroid of $\Delta H\pi L$............................ (7).

The equations to the circles DEF, D′E′F′, are

$$\alpha\beta\gamma + \ldots + \ldots = (a\alpha + \ldots + \ldots)(\lambda'\alpha + \mu'\beta + \nu'\gamma),$$

where $\lambda' \text{ in } (1^\circ) = a^4 c \cos B \left[b^2\nu^4 + 2(c^2 - a^2)\lambda^2\right]/P$,

$$\text{in } (2^\circ) = a^4 b \cos C \left[c^2\nu^4 + 2(b^2 - a^2)\lambda^2\right]/P,$$

where $P = 2k\, a^3 b^3 c^3 (1 + \cos A \cos B \cos C)$.

The conic round the two triangles may be called the "cosine" conic of the triangle (because $DD' \propto \cos A$); its equation is

$$4\left[\alpha^2 a^4 \cot B \cot C + \ldots + \ldots\right] = \beta\gamma \cot A \operatorname{cosec} A (a^4 - b^4 - c^4 + 6b^2c^2) + \ldots + \ldots.$$

(10045.) In the case of the medial triangle, as in (A) we have

$$\alpha + \alpha' = (a\alpha + b\beta + c\gamma)/2a,$$

therefore $2a\alpha' = b\beta + c\gamma - a\alpha$, and $a\alpha = b\beta' + c\gamma'$ ....(B, B′).

From (B′), coördinates of Ω_1, Ω_1' are respectively

$$b^2(c^2+a^2)/a,\ \ldots,\ \ldots,\quad c^2(a^2+b^2)/a,\ \ldots,\ \ldots;$$

$\Omega\Omega_1$, $\Omega'\Omega_1'$ are given by

$$\alpha ab\,(c^2-a^2)+\ldots+\ldots=0,\quad \alpha ac^2(a^2-b^2)+\ldots+\ldots=0,$$

which pass through the centroid.

$\Omega\Omega_1'$, $\Omega'\Omega_1$ are given by

$$a^3\alpha b^2\mathrm{N}+\ldots+\ldots=0,\qquad a^3\alpha c^2(\mathrm{M})+\ldots+\ldots=0,$$

(where $\mathrm{L}\equiv a^4-b^2c^2$, $\mathrm{M}\equiv b^4-c^2a^2$, $\mathrm{N}\equiv c^4-a^2b^2$),

which pass through $\qquad a^3\alpha=b^3\beta=c^3\gamma\ (\mathrm{T})$ ....(1);

V is $a(b^2+c^2)$, ..., ..., and V′ [by (B′)] is $(\lambda^2+b^2c^2)/a$, ..., ..., therefore VV′ is $a^3\alpha(b^2-c^2)+\ldots+\ldots=0$, which passes through both G and T.

The α of V is $2\Delta a(b^2+c^2)/2\lambda^2$, α' of V′ is $2\Delta(\lambda^2+b^2c^2)/4a\lambda^2$,

therefore $$\frac{2\alpha'+\alpha}{3}=\frac{2\Delta}{3a}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

hence G is a point of trisection of VV′.

By (B′) coordinates of L′ are as $(b^2+c^2)/a,\ c^2/b,\ b^2/c,$

M′ ,, $c^2/a,\ (c^2+a^2)/b,\ a^2/c,$

N′ ,, $b^2/a,\ a^2/b,\ (a^2+b^2)/c.$

Therefore join of LL′ is $\quad a\alpha(b^2-c^2)+bc^2\beta-b^2c\gamma=0,$

of MM′ $\quad -c^2a\alpha+b\beta(c^2-a^2)+ca^2\gamma=0,$

of NN′ $\quad ab^2\alpha-a^2b\beta+c\gamma(a^2-b^2)=0;$

these evidently meet in G. The join of AL′ is $b^3\beta=c^3\gamma$ ....(3), therefore AL′, BM′, CN′ cointersect in T.

The equation to the B. circle of D′E′F′ ["T.R. circle," xx. and (B)] is $abc\left[(b\beta+c\gamma-a\alpha)^2/a^2+\ldots+\ldots\right]=a^3\left[a^2\alpha^2-(b\beta-c\gamma)^2\right]/bc+\ldots+\ldots,$

i.e., $\quad a^2\alpha^2\left[\lambda^2+\nu^4-2a^4\right]+\ldots+\ldots=2bc\beta\gamma\left[a^2k-b^2c^2\right]+\ldots+\ldots$

Putting this under the form $\alpha\beta\gamma+\ldots+\ldots=(p\alpha+\ldots+\ldots)(a\alpha+\ldots+\ldots)$, we readily get $\quad 4abckp=a(\lambda^4+\nu^4-2a^4)=a(\mathrm{Q}-2a^4)$ ....(4).

The equation to the B. ellipse of ABC is

$$\frac{\alpha^2}{a^2}+\ldots+\ldots=2\frac{\beta\gamma}{bc}+\ldots+\ldots;$$

hence, by (B), the equation to the B. ellipse of D′E′F′ is

$$a^2\alpha^2\left[k'^2-4/b^2c^2\right]+\ldots+\ldots=2bc\beta\gamma\left[k'^2-2k'/a^2\right]+\ldots+\ldots,$$

where $\qquad k'=1/a^2+1/b^2+1/c^2$ ....(5).

The equation to the B. circle of the pedal triangle is got by means of (A′), $a'b'c'\left[(-\alpha\cos\mathrm{A}+\beta\cos\mathrm{B}+\gamma\cos\mathrm{C})^2+(\ )^2+(\ )^2\right]$

$$=a'^3\left[\alpha^2\cos^2\mathrm{A}-(\beta\cos\mathrm{B}-\gamma\cos\mathrm{C})^2\right]+\ldots+\ldots;$$

i.e., $\quad \alpha^2\cos^2\mathrm{A}\left[b'^3+c'^3-a'^3+3a'b'c'\right]+\ldots+\ldots$

$$=2\beta\gamma\cos\mathrm{B}\cos\mathrm{C}\left[a'^3+a'b'c'\right]+\ldots+\ldots,$$

where for a', b', c' we can write sin 2A, sin 2B, sin 2C.

In the manner of (A), we find the transformation for the in-circle tri-

angle to be $\quad \alpha \cos \frac{1}{2}A = \beta' \cos \frac{1}{2}C + \gamma' \cos \frac{1}{2}B, \ldots, \ldots, \qquad \ldots\ldots\ldots\ldots(C),$
and $\quad 2\alpha' \cos \frac{1}{2}B \cos \frac{1}{2}C = -\alpha \cos^2 \frac{1}{2}A + \beta \cos^2 \frac{1}{2}B + \gamma \cos^2 \frac{1}{2}C, \ldots, \qquad \ldots(C'),$
with $\qquad a' = 2(s-a) \sin \frac{1}{2}A.$
The coordinates of the B. points are, by (C),
$| \; (a'^2 b' \cos \frac{1}{2}C + b'^2 c' \cos \frac{1}{2}B)/\cos \frac{1}{2}A, \quad | \; (b'c'^2 \cos \frac{1}{2}C + c'a'^2 \cos \frac{1}{2}B)/\cos \frac{1}{2}A,$
these reduce to the given expressions.

(10101.) The sides of $\alpha\beta\gamma$ are $a \cos A$, $b \cos B$, $c \cos C$, and the angles are $\pi - 2A$, $\pi - 2B$, $\pi - 2C$, therefore coordinates of α' (referred to $\alpha\beta\gamma$) are $| \; 0, \; -\cos 2C, \; -\cos 2B$; hence (by A) the coordinates referred to ABC are as $2\cos(B-C)$, $-\cos 2B/\cos B$, $-\cos 2C/\cos C$; hence equation to $A\alpha$ is $\beta \cos B/\cos 2B = \gamma \cos C/\cos 2C$, and the three lines cointersect in

$$\cos 2A / \cos A, \; \ldots, \; \ldots, \; (\pi_3),$$

and as the CONG line is $\quad a\alpha \cos A \sin(B-C) + \ldots + \ldots = 0 \quad \ldots\ldots\ldots\ldots(1),$
we see that π_3 is on it.

The equation to $O\alpha'$ is

$$\alpha \sin 2A \sin(B-C) + \beta \cos B \cos 2B - \gamma \cos C \cos 2C = 0,$$

therefore equation to AL is $\beta \cos B \cos 2B = \gamma \cos C \cos 2C \quad \ldots\ldots\ldots\ldots(2),$
hence AL, BM, CN meet in π_4.*

Now, as the join of the orthocentres in question is

$$a\alpha \cos A \cos 2A \sin(B-C) + \ldots + \ldots = 0,$$

we see that π_4 lies on it.

If $\alpha_1, \alpha_2, \ldots \alpha_{n+1}$ are proportional to the α-coordinates of the S. points and B. points of the primitive and first, second, ... n^{th} medial triangles, we have (by B') $\quad \alpha_1 \; | \; a, \quad \alpha_2 \; | \; (k - a^2)/a, \quad \alpha_3 \; | \; (k + a^2)/a,$

$$\alpha_4 \; | \; (3k - a^2)/a, \quad \alpha_5 \; | \; (5k + a^2)/a, \text{ and so on,}$$

hence (for S. points) $a\alpha_{n+1} = pk + (-1)^n a^2$, where $3p \equiv 2^n + (-1)^n$;
For the B. points $\qquad a\alpha_1 \; | \; a^2c^2, \qquad | \; a^2b^2,$

$$a\alpha_2 \; | \; \lambda^2 - a^2c^2, \qquad | \; \lambda^2 - a^2b^2,$$

and so on, as before, therefore

$$a\alpha_{n+1} \; | \; p\lambda^2 + (-1)^n a^2c^2, \qquad | \; p\lambda^2 + (-1)^n a^2b^2.$$

(10139.) The "T. R." triangles DEF, D'E'F' are got by the intersection of the "T. R." circle with the sides of ABC (*cf. Quar. Jour. of Math.*, Vol. XIX., No. 76), $\angle F = A, \quad \angle D = B, \quad \angle E = C$;
hence, by the method of (A), we get

$$\left.\begin{aligned} \alpha \sin B &= \alpha' \sin(B+\omega) + \gamma' \sin\omega \\ \beta \sin C &= \beta' \sin(C+\omega) + \alpha' \sin\omega \\ \gamma \sin A &= \gamma' \sin(A+\omega) + \beta' \sin\omega \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(D),$$

whence $\qquad \alpha' \; | \; a(a^2+b^2)(b^2+c^2) + \beta abc^2 - \gamma ac(a^2+b^2) \ldots\ldots\ldots\ldots\ldots(D').$

---

* It is interesting to notice that this is the inverse of the point P' of my "Isoscelians" (*London Mathematical Society's Proceedings*, Vol. XIX., pp. 163—170), which plays an important part in the construction of the "S. T. A." circle.

Now it is known that Ω, K are the B. points of DEF,
and K, Ω' ,, D'E'F'.

To find the S. point of DEF (k_1), we have (by D)

$\alpha \sin B \mid a \sin B \cos \omega + \sin \omega\,(a \cos B + C)$, whence $\alpha \mid a\,(a^2+2c^2)$.

Similarly for k_2 (ΔD'E'F'), $\alpha \mid a\,(a^2+2b^2)$;
the mid-point of $k_1 k_2$ is hence seen to be K.

For the centroid G_1 of DEF, $\alpha \mid (2a^2+c^2)/a$,
and for ,, G_2 of D'E'F', $\alpha \mid (2a^2+b^2)/a$.

Putting the coordinates of K, Ω, k_1, and of K, Ω', k_2, in the equation

$$a\beta\gamma + \ldots + \ldots = (p\alpha + \ldots + \ldots)(a\alpha + \ldots + \ldots),$$

we get $p = bc\,(2b^2+c^2)/k^2, \quad = bc\,(b^2+2c^2)/k^2$ respectively .... (1).
Of these circles, of course, the radical axis is the circum-Brocardal-axis of ABC.

The equation to $k_1 k_2$ is $bc\alpha\,(3a^2-k) + \ldots + \ldots = 0$, which passes through K and is parallel to $bc\alpha + \ldots + \ldots = 0$, and is therefore perpendicular to the circum-Br.-axis (and touches the B. circle) of ABC ....(2).
Hence we get an easy construction for k_1, k_2.

The mid-point of G_1G_2 is $\mid (3a^2+k)/a, \ldots, \ldots$, which is also the mid-point of KG, and lies on the S. point axis.

The transformation for the *diametral* triangles turns upon the fact that

$$\alpha + \alpha' = 2R \cos A \qquad \text{.............................(E)};$$

whence, taking the equation to the Brocard circle under the form

$$k\,(a\beta\gamma + \ldots + \ldots) = (a\alpha + \ldots)(bc\alpha + \ldots),$$

we get $k\,(a\beta\gamma + \ldots) + (a\alpha + \ldots + \ldots)(bc\alpha + \ldots) = 2R \cot \omega\,(a\alpha + \ldots)^2$,
which reduces to

$$16\Delta^2 k\,(a\beta\gamma + \ldots) = (a\alpha + \ldots)\left[(\nu^4 + 2L)\,bc\alpha + \ldots + \ldots\right] \quad \text{.........(4)}.$$

Hence the radical axis of the two B. circles is

$$bc\alpha + \ldots = R \cot \omega\,(a\alpha + \ldots),$$

Since, from symmetry, $O\Omega = O\Omega' = O\Omega_2 = O\Omega_2'$ (Ω_2, Ω_2' the B. points of A'B'C'), the circle round the four B. points is

$$a\beta\gamma + \ldots = P\,(a\alpha + b\beta + \ldots)^2;$$

to find P, since Ω is on the circle, we have $abc\lambda^2 = P\lambda^4$, and therefore the equation becomes $\lambda^2\,(a\beta\gamma + \ldots) = abc\,(a\alpha + \ldots)^2$ ....(5).

The coordinates of M, N are

$$(a \cot A,\ 0,\ a \cos B \cot C), \quad (a \cot A,\ a \cot B \cos C,\ 0),$$

therefore equations to MB, NC are

$$\gamma \cot A = \alpha \cos B \cot C, \quad \alpha \cot B \cos C = \beta \cot A;$$

these intersect in $\alpha/\cot A = \beta/\cot B \cos C = \gamma/\cos B \cot C$,
which lies on $\beta b = \gamma c$, therefore the truth of (6).

Making use of (E), we get the diametral B. ellipse from the equation

$$\left(\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} - \cot \omega\right)^2 = 4\left\{\frac{\beta\gamma}{bc} + \frac{\gamma\alpha}{ca} + \frac{\alpha\beta}{ab}\right\},$$

or

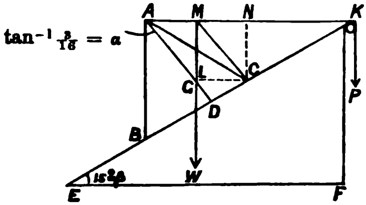
$$\left\{\frac{\alpha}{a}\,(\nu^4 + 2L) + \ldots\right\}^2 = 1024\Delta^4\left\{\frac{\beta\gamma}{bc} + \ldots\right\},$$

or

$$\left\{bc\alpha\,(\nu^4 + 2L) + \ldots\right\}^2 = 1024\,abc\Delta^4\,(a\beta\gamma + \ldots) \quad \text{...........(7)}.$$

8926. (Professor ASUTOSH MUKHOPÂDHYÂY, M.A., F.R.A.S.)—A right circular cone, the semi-vertical angle of which is $\tan^{-1}\frac{3}{16}$, is placed with its base on a smooth plane inclined at 75° to the vertical; to the vertex of the cone is attached a fine string, which, passing over a pulley, on the inclined plane, at the same height as the vertex, sustains a heavy particle. If the system is in limiting equilibrium, show that the ratio of the weights of the cone and the particle is such that the slightest increase in the weight of the particle would cause the cone to turn about the highest point of the base as well as to slide.

Solution by G. G. STORR, M.A.; Rev. T. GALLIERS, M.A.; *and others.*



Let $AD = h$, $GD = \frac{1}{4}h$,
AK horizontal, GM vertical,
CL ,, CN ,,
Now
$$CD : CM = h\tan\alpha : AG\sin\beta$$
$$= 1 : 4\sin\beta,$$
and $DK : AK = \cos\beta : 1$,
$\therefore$ $CD : AM = DK : AK$.
Since $2\sin 2\beta = 1$.

Hence CM is perpendicular to BC, so that the reaction of the plane passes through C, and therefore the cone is on the point of turning about C.

Also (moments about C) $P \cdot CN = W \cdot CL$,

or $$P(\cos\beta - \tan\alpha\sin\beta) = W(\tan\alpha\cos\beta + \tfrac{1}{4}\sin\beta) \quad \text{............(1)}.$$

If the cone be on the point of sliding, we have
$$P\cos\beta = W\sin\beta \quad \text{..............................(2)},$$
and equations (1) and (2) give the same value for P : W, when
$$3\sin 2\beta = 8\tan\alpha \equiv \tfrac{3}{2} \quad \text{or} \quad 2\beta = 30°,$$
which is the case.

10027. (Professor ABINASH CHANDRA BASÛ.)—If
$$y^2 + yz + z^2 = a^2, \quad z^2 + zx + x^2 = b^2, \quad x^2 + xy + y^2 = c^2,$$
a, b, c being the sides of a triangle; find (1) the value of $xy + yz + zx$; and (2) show how to solve the set of equations.

Solution by J. D. H. DICKSON, M.A.; R. KNOWLES, B.A.; *and others.*

If the triangle is ABC, and x, y, z are the distances of a point O from A, B, C respectively, O is the point where $x + y + z$ is a minimum. For, if B, C be the foci of an ellipse through O, AO ($= x$) is shortest when AO is normal to the ellipse, and in this case $\angle AOB = \angle AOC$, and similarly $= \angle BOC$. From these we get
$$\cos AOB = \frac{x^2 + y^2 - c^2}{2xy} = -\tfrac{1}{2}, \text{ whence } x^2 + xy + y^2 = c^2, \text{ \&c.}$$

Hence, if Δ = area of triangle ABC,

$$(xy+yz+zx)\sin \text{AOB} = 2\Delta, \text{ therefore } xy+yz+zx = \tfrac{4}{3}\sqrt{3}\,\Delta;$$

Again, $$a^2+b^2+c^2+4\sqrt{3}\,\Delta = 2x^2+2y^2+2z^2+4xy+4yz+4zx$$
$$= 2(x+y+z)^2 = 2\text{P}^2 \text{ (say)},$$

therefore $$\text{P}^2-a^2 = x^2+2xy+yz+2zx = \text{P}x+\tfrac{4}{3}\sqrt{3}\,\Delta,$$

whence $\text{P}x = \frac{1}{2}(-a^2+b^2+c^2)+\frac{2}{3}\Delta\sqrt{3}$; and similarly for y, z.

10065. (Professor Matz, M.A.)—A man, standing on a plain, observes a row of equidistant pillars, the tenth and seventeenth of which subtend the same angles as they would if they stood in the position of the first and were respectively one-half and one-third of the height; show that, neglecting the height of the eye, the line of the pillars is inclined to the line drawn to the first at an angle whose cosine is $5/(168)^{\frac{1}{2}}$ or $\frac{5}{13}$ nearly.

Solution by D. Biddle; C. Morgan, M.A.; *and others.*

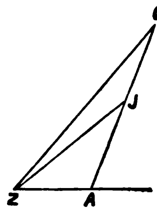
Let the distance from pillar to pillar be unity, and AZ = x, then

$$\text{AJ} = 9, \quad \text{AQ} = 16, \quad \text{JZ} = 2x, \quad \text{QZ} = 3x,$$

and $$\cos \text{QAZ} = (256+x^2-9x^2)/(32x)$$
$$= (81+x^2-4x^2)/(18x),$$

whence $$x = (42)^{\frac{1}{2}},$$

and $$\cos \text{QAZ} = -5/(168)^{\frac{1}{2}}.$$



10034. (Professor Morley.)—Prove that, of the four focal circles of a circular cubic or bicircular quartic, any two are orthogonal, and the radii are connected by the relation $\Sigma(\mu^{-2}) = 0$.

Solution by W. J. C. Sharp, M.A.; Sarah Marks, B Sc.; *and others.*

If $x^2+y^2-2hx-2ky+c=0$ and $x^2+y^2-2h'x-2k'y+c'=0$ cut orthogonally $2hh'+2kk'=c+c'$, we have

$$2\frac{ha^2}{c}\cdot\frac{h'a^2}{c'}+2\frac{ka^2}{c}\cdot\frac{k'a^2}{c'} = \frac{a^4}{c}\cdot\frac{a^4}{c'},$$

therefore the following inverses of these circles also cut orthogonally

$$x^2+y^2-\frac{2ha^2}{c}x-\frac{2ka^2}{c}y+\frac{a^4}{c}=0 \quad\text{and}\quad x^2+y^2-\frac{2h'a^2}{c'}x-\frac{2k'a^2}{c'}y+\frac{a^4}{c}=0.$$

I have shown (Vol. xxxv., p. 47) that the equation

$$x(x^2+y^2) = ax^2+2bxy+2fy+2gx$$

will represent any circular cubic, and that the corresponding equation to the focal circles is

$$A^2(x^2+y^2)+2A^3x+2A(f-bA)-2gA^2+2bfA-2f^2=0,$$

where A is any root of $A^4+aA^3-2gA^2=(f-bA)^2$........................(i.).

If, then, A_1 and A_2 be any two roots of (i.), the corresponding focal circles will cut orthogonally, if

$$2A_1A_2+2\left(\frac{f}{A_1}-b\right)\left(\frac{f}{A_2}-b\right)+4g-2bf\left(\frac{1}{A_1}+\frac{1}{A_2}\right)+2f^2\left(\frac{1}{A_1^2}+\frac{1}{A_2^2}\right)=0,$$

i.e., if
$$A_1A_2+\left(\frac{f}{A_1}-b\right)\left(\frac{f}{A_2}-b\right)+2g+\left(\frac{f}{A_1}-b\right)^2$$
$$+b\left(\frac{f}{A_1}-b\right)+\left(\frac{f}{A_2}-b\right)^2+b\left(\frac{f}{A_2}-b\right)=0,$$

or, since A_1 and A_2 are roots of (i.), if

$$A_1^2+A_1A_2+A_2^2+a(A_1+A_2)-2g+\frac{f^2}{A_1A_2}-b^2=0 \quad\text{.........(ii.)};$$

but, by (1), $$A_1^3+aA_1^2-2gA_1=\frac{1}{A_1}(f-bA_1)^2,$$

and $$A_2^3+aA_2^2-2gA_2=\frac{1}{A_2}(f-bA_2)^2,$$

therefore $$(A_1-A_2)\left\{A_1^2+A_1A_2+A_2^2+a(A_1+A_2)-2g+\frac{f^2}{A_1A_2}-b^2\right\}=0,$$

and therefore, if A_1 and A_2 are unequal, (ii.) is satisfied, and the focal circles cut orthogonally. Hence the theorem holds for circular cubics, and since every bicircular quartic is the inverse of a circular cubic, and the focal circles of the one are the inverses of those of the other (*Reprint*, Vol. xxxv., p. 47), the theorem is also, by what was proved above, true for bicircular quartics.

Since the four circles are mutually orthogonal, it follows that $\Sigma(\mu^{-2})=0$, and hence that one, and therefore from (i.) two, are imaginary (*Proceedings of the London Mathematical Society*, Vol. xix., p. 459).

9902. (Professor LAISANT.)—On donne une circonférence Δ, un point fixe O et un axe fixe OX. Soit M un point quelconque de Δ. (1) Soient N un point tel que *angle* NOX $=\frac{1}{2}$MOX, ON $=(a\,.\,\text{OM})^{\frac{1}{2}}$, a étant une constante; démontrer que N décrit un ovale de Cassini. (2) Le lieu d'un point P tel que *angle* POX = 2MOX, OP = OM^2/a est un ovale de Descartes.

Solution by R. F. DAVIS, M.A.

If upon the bisector of the angle A of a triangle ABC (Fig. 1), points P, Q be taken such that $AP^2=AQ^2=AB\,.\,AC$; then

$$BP\,.\,BQ=BC\,.\,AB, \quad\text{and}\quad CP\,.\,CQ=BC\,.\,AC.$$

Let D be the mid-point of BC; E, F the extremities of the diameter of the circumcircle through D, F and A lying on the same side of BC.

Then, since the triangles ABP, APC are similar, and the angle BPC $= \pi - \frac{1}{2}A$, the points B, P, C, Q obviously lie on a circle centre F. Hence, if CM be perpendicular on AD, CP . CQ = 2CM . CF.

But CM : AC = CD : CF, therefore CP . CQ = 2CD . AC = BC . AC.

[The trigonometrical interpretation of the above is that the product of the two quantities $b + c \pm 2(bc)^{\frac{1}{2}} \cos \frac{1}{2}A = a^2$. The points P, Q seem to possess many interesting properties, one of which was used in the geometrical investigation of the distance between the in- and nine-point centres of a triangle; see our Vol. xxx., p. 99, and MILNE's *Companion*, pp. 187, 188.]

(See now Fig. 2.)

Let D be the centre of Δ upon which M lies. Upon OX take OA = a. Let OC, OR bisect the angles DOX, DOM, respectively, and

$$OC^2 = OA . OD, \quad OR^2 = OD . OM.$$

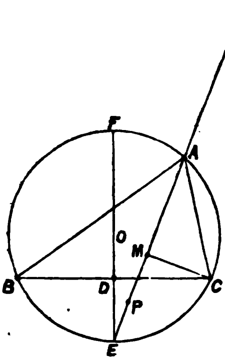


Fig. 1.

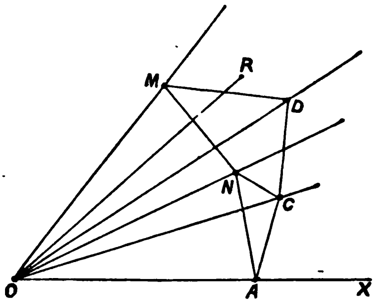


Fig. 2.

Then $\quad ON^2 : OR^2 = OA . OM : OD . OM = OA : OD$

$$= OC^2 : OD^2 = \text{constant},$$

and the angle RON $= \frac{1}{2}(AOM - DOM) = \frac{1}{2}AOD = COD$;

hence the locus of N is similar to that of R. But the locus of R is a Cassinian oval. For, by the preceding proposition, if RO, DO be produced to R′, D′, so that OR′ = OR, OD′ = OD, then

$$DR . DR' = OD . DM = \text{constant}, \text{ that is, } DR . D'R \text{ is constant.}$$

The locus of N is therefore also a Cassinian oval, whose foci are C, C′.

If in the above figure DM be no longer constant, but CN be constant, then the locus of R is a circle centre D (for CN : DR = const.).

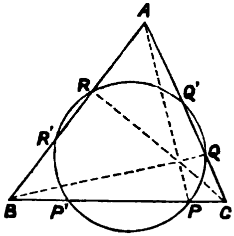
But $\quad DR . D'R = OD . DM, \quad$ therefore $D'R \propto DM$.

Also $\quad DR^2 + D'R^2 = 2(OR^2 + OD^2), \quad$ or $DM^2 - l . OM = \text{const.}$

Hence the locus of M (and therefore of P) is a Cartesian oval.

9982. (Professor STEGGALL.)—If a circle intersect the sides of a triangle ABC in PP′, QQ′, RR′, and if AP, BQ, CR are concurrent, so also are AP′, BQ′, CR′.

Solution by J. J. BARNIVILLE; R. H. W. WHAPHAM, B.A.; *and others.*



$$BP \cdot CQ \cdot AR / CP \cdot AQ \cdot BR = 1,$$

therefore $BP' \cdot CQ' \cdot AR' / CP' \cdot AQ' \cdot BR'$

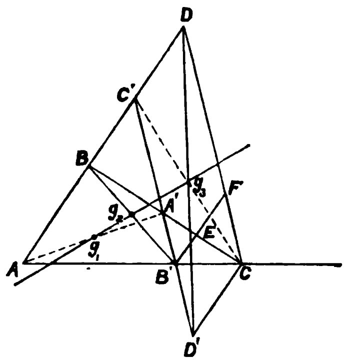
$$= \frac{BP \cdot BP' \cdot CQ \cdot CQ' \cdot AR \cdot AR'}{CP \cdot CP \cdot AQ \cdot AQ' \cdot BR \cdot BR'} = 1,$$

therefore AP′, BQ, CR′ are concurrent.

[This theorem is true if, instead of a circle, we have any conic intersecting the sides of the triangle in question.]

10076. (Professor GENESE, M.A.)—ABA′B′ is any quadrilateral; AB′, BA′ meet at C; from C are drawn parallels to AB, A′B′, meeting A′B′, AB respectively in D′, D. Prove that B′A′ : A′D′ :: BA : AD. Hence show that the well-known theorem about the middle points of the diagonals of a completed quadrilateral, is a particular case of the theorem, that the mass-centre of two particles with uniform rectilineal velocities describes a straight line.

Solution by R. H. W. WHAPHAM, B.A.; Professor SARKAR; *and others.*



Draw B′EF parallel to AB or CD′, therefore

$$B'A' : A'D' = B'E : D'C$$
$$= B'E : B'F = BA : AD,$$

Now suppose two equal masses at A and A′, their centre of mass is at g_1, the mid-point of AA′; let these masses move with uniform velocities proportional to AB and A′B′ respectively in the directions AD and A′D′. Then, when the masses are at B and B′, their centre of mass will be at g_2, the mid-point of BB′. Also, since

$$B'A' : A'D' = BA : AD,$$

the masses will arrive at D′ and D at the same time, and their centre of mass will be at g_3, the mid-point of DD′; but g_3, the mid-poin of DD′, is also the mid-point of CC′. Therefore g_1, g_2, g_3 are in a line.

9790. (W. J. C. SHARP, M.A.) — If u be a rational and integral symmetrical function of $x_1, x_2 \dots x_n$, show that

$$x_r^p \frac{du}{dx_r} - x_s^p \frac{du}{dx_s} \text{ and } x_s^p \frac{du}{dx_r} - x_r^p \frac{du}{dx_s}$$

are divisible by $x_r - x_s$ for all positive integral values of p, r, and s.

Solution by D. EDWARDES; Prof. BEYENS; *and others.*

Let $x_1, x_2 \dots x_n$ be the roots of $p_0 x^n - p_1 x^{n-1} + \dots = 0$, and let

$$p_0 x^n - p_1 x^{n-1} + \dots = (x - x_r)(p_0 x^{n-1} - p_1' x^{n-2} + p_2' x^{n-3} - \dots),$$

then $\quad p_k = p_k' + x_r p_{k-1}', \quad$ therefore $\dfrac{dp_k}{dx_r} = p_{k-1}'$.

Let u be a rational, integral, and symmetrical function of $x_1, x_2 \dots x_n$, and let $\quad u = f(p_0, p_1, p_2 \dots p_n)$,

Then $$\frac{du}{dx_r} = \frac{df}{dp_0}\frac{dp_0}{dx_r} + \frac{df}{dp_1}\frac{dp_1}{dx_r} + \&c.,$$

hence $$\frac{du}{dx_r} = p_0 \frac{df}{dp_1} + p_1' \frac{df}{dp_2} + p_2' \frac{df}{dp_3} + \dots.$$

Similarly, if $p_0 x^n - p_1 x^{n-1} + \dots = (x - x_s)(p_0 x^{n-1} - q_1 x^{n-2} + q_2 x^{n-3} \dots)$,

we have $$\frac{du}{dx_s} = p_0 \frac{df}{dp_1} + q_1 \frac{df}{dp_2} + q_2 \frac{df}{dp_3} + \dots,$$

therefore $$x_r^p \frac{du}{dx_r} - x_s^p \frac{du}{dx_s} = p_0 \frac{df}{dp_1}(x_r^p - x_s^p) + \frac{df}{dp_2}(p_1' x_r^p - q_1 x_s^p) + \&c.$$

Now $$p_k' + x_r p_{k-1}' = q_k + x_s q_{k-1} = p_k,$$

hence $$x_r^p p_k' - x_s^p q_k = p_k (x_r^p - x_s^p) - (x_r^{p+1} p_{k-1}' - x_s^{p+1} q_{k-1});$$

and $p_0' = q_0 = p_0$, hence $x_r^p p_k' - x_s^p q_k$ is divisible by $x_r - x_s$, and therefore also so is $$x_r^p \frac{du}{dx_r} - x_s^p \frac{du}{dx_s}.$$

Again, $\quad x^p p_k' - x_r^p q_k = p_k (x_s^p - x_r^p) - (x_r x_s^p p_{k-1}' - x_s x_r^p q_{k-1})$,

and therefore (since $p_0' = q_0 = p_0$), $x_s^p p_h' - x_r^p q_h$ is divisible by $x_s - x_r$.

9366. (ASPARAGUS.)—A chord PQ of an ellipse is normal at P, CZ meets this chord at right angles in Z, and meets the ellipse in D; prove that the difference of the eccentric angles of P and Q is $2\tan^{-1}$ (CD/CZ); and its maximum value is $4\tan^{-1}(b/a)$, when PQ is the diameter of curvature at P.

Solution by Rev. T. GALLIERS, M.A.; G. G. STORR, M.A.; *and others.*

If P, Q be ϕ_1, ϕ_2, then, since PQ is normal at P, we have

$$\tan \tfrac{1}{2}(\phi_1 - \phi_2) = (a^2 \sin^2 \phi_1 + b^2 \cos^2 \phi_1) / \left\{(a^2 - b^2)\sin\phi_1 \cos\phi_1\right\} \quad \dots (1),$$

and $\quad \{b^2\cos^2\frac{1}{2}(\phi_1+\phi_2)+a^2\sin^2\frac{1}{2}(\phi_1+\phi_2\}/(a^2\sin^2\phi_1+b^2\cos^2\phi_1)$

$$= a^2b^2\cos^2\tfrac{1}{2}(\phi_1-\phi_2)\,\{(a^2-b^2)^2\sin^2\phi_1\cos^2\phi_1\} \;\ldots\ldots\ldots\ldots\; (2).$$

Again, $\quad CD = (a^2\sin^2\phi_1+b^2\cos^2\phi_1)^{\frac{1}{2}},$

and $\quad CZ = (a^2-b^2)\sin\phi_1\cos\phi_1/(a^2\sin^2\phi_1+b^2\cos^2\phi_1)^{\frac{1}{2}}.$

Hence $\quad \phi_1-\phi_2 = 2\tan^{-1}(CD/CZ),$

and $\quad CD/CZ = (a^2\tan\phi_1+b^2\cot\phi_1)/(a^2-b^2),$

This last is a *minimum* when $\tan\phi_1 = b/a$, *i.e.*, when $\phi_1-\phi_2 = 4\tan^{-1}b/a$.

By the aid of (2) we may show that

$$PQ = \frac{2ab}{a^2-b^2}\sin\tfrac{1}{2}(\phi_1-\phi_2)\;\cos\tfrac{1}{2}(\phi_1-\phi_2)\;\frac{(a^2\sin^2\phi_1+b^2\cos^2\phi_1)^{\frac{1}{2}}}{\sin\phi_1\cos\phi_1}$$

$$= 4a^2b^2\sqrt{2}/(a^2+b^2)^{\frac{3}{2}} \text{ when } \phi_1-\phi_2 = 4\tan^{-1}b/a,$$

also the diameter of curvature at P

$$= 2(a^2\sin^2\phi_1+b^2\cos^2\phi_1)^{\frac{3}{2}}/(ab)$$

$$= 4a^2b^2\sqrt{2}/(a^2+b^2)^{\frac{3}{2}}, \text{ when } \phi_1 = \tan^{-1}b/a,$$

which proves the last part of the Question.

9151. (D. EDWARDES.)—Prove that (1)

$$\int_{-b^2}^{-c^2}\int_{-a^2}^{-b^2}\frac{uv(u-v)\,du\,dv}{\{-(a^2+u)(b^2+u)(c^2+u)(a^2+v)(b^2+v)(c^2+v)\}^{\frac{1}{2}}}$$

$$= -\tfrac{2}{3}\pi(a^2b^2+b^2c^2+c^2a^2),$$

and (2) deduce Legendre's theorem $EF'+E'F-FF' = \frac{1}{2}\pi$.

Solution by the PROPOSER.

Let dS be an element of the surface of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2} = 1$, p the central perpendicular on the tangent plane. Let $V = \frac{1}{2}\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\right)$. Then, $\dfrac{dV}{dn}$ denoting differentiation in the direction of the normal at x, y, z, we have $\dfrac{dV}{dn} = p\left(\dfrac{x}{a^2}\dfrac{dV}{dx}+\dfrac{y}{b^2}\dfrac{dV}{dy}+\dfrac{z}{c^2}\dfrac{dV}{dz}\right) = p\left(\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}+\dfrac{z^2}{c^4}\right) = \dfrac{1}{p}$. Hence $\displaystyle\int\frac{dS}{p}$ (taken over the surface) $= \displaystyle\int\frac{dV}{dn}dS$, which, by GREEN'S theorem $= \displaystyle\iiint\left(\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}\right)dx\,dy\,dz$ (taken throughout the volume) $= \displaystyle\iiint\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)dx\,dy\,dz = \frac{4}{3}\pi\,.\,\frac{a^2b^2+b^2c^2+c^2a^2}{abc}$.

Hence $\int \frac{dS}{p}$ (taken over $\frac{1}{8}$th part of surface) $= \frac{\pi}{6abc}(a^2b^2+b^2c^2+c^2a^2)$.

Now (see FERRERS' *Spherical Harmonics*, page 129),

$$\frac{dS}{p} = \frac{1}{4} \cdot \frac{(v-u)\,uv\,du\,dv}{\left\{-(a^2+u)(b^2+u)(c^2+u)(a^2+v)(b^2+v)(c^2+v)\right\}^{\frac{1}{2}}}.$$

Hence, integrating over $\frac{1}{8}$th part of the surface,

$$\int_{-b^2}^{-c^2}\int_{-a^2}^{-b^2} \frac{uv\,(u-v)\,du\,dv}{\left\{-(a^2+u)(b^2+u)(c^2+u)(a^2+v)(b^2+v)(c^2+v)\right\}^{\frac{1}{2}}}$$

$$= -\tfrac{2}{3}\pi\,(a^2b^2+b^2c^2+c^2a^2).$$

Put $u = -c^2\sin^2\phi - b^2\cos^2\phi$, $v = -b^2\sin^2\theta - a^2\cos^2\theta$. The integral becomes

$$\frac{4}{a^2-c^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{A^2B-AB^2}{(1-k^2\sin^2\theta)^{\frac{1}{2}}(1-k'^2\cos^2\phi)^{\frac{1}{2}}}\,d\theta\,d\phi,$$

where

$A = a^2-(a^2-b^2)\sin^2\theta$, $B = c^2+(b^2-c^2)\cos^2\phi$, $k^2 = \frac{a^2-b^2}{a^2-c^2}$, $k'^2 = \frac{b^2-c^2}{a^2-c^2}$,

so that $k^2+k'^2 = 1$.

Now (see HYMER's *Integral Calculus*, 2nd edition, page 191),

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta}{\Delta(\theta)}\,d\theta = -\frac{1}{3k^4}\left\{2(1+k^2)\,E-(2+k^2)\,F\right\},$$

and

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta}{\Delta(\theta)}\,d\theta = -\frac{1}{k^2}(E-F);$$

and hence, if I denote the integral above,

$$\tfrac{1}{4}(a^2-c^2)\,I$$

$$= \left[a^2F'-(a^2-c^2)E'\right]\left[\tfrac{2}{3}(a^2-c^2)(a^2+b^2+c^2)E+\tfrac{1}{3}(2c^4+b^2c^2-a^2b^2+a^2c^2)F\right]$$
$$-\left[(a^2-c^2)E+c^2F\right]\left[-\tfrac{2}{3}(a^2-c^2)(a^2+b^2+c^2)E'+\tfrac{1}{3}(2a^4+a^2c^2+a^2b^2-b^2c^2)F'\right],$$

and the right-hand side reduces to

$$\tfrac{1}{3}(a^2-c^2)(a^2b^2+b^2c^2+c^2a^2)(EF'+E'F-FF'),$$

so that

$$EF'+E'F-FF' = \tfrac{1}{2}\pi.$$

9998. (H. L. ORCHARD, M.A., B.Sc.)—Show that the roots of the equation $\Sigma \equiv x^6+12x^5+14x^4-140x^3+69x^2+128x-84 = 0$ are

$$-1,\ 2,\ -7,\ 1,\ \tfrac{1}{2}(-7\pm\sqrt{73}).$$

Solution by J. BARNIVILLE, J. D. H. DICKSON, *and others.*

$$\Sigma \equiv (x^3+6x^2-11x-4)^2-(2x-10)^2$$
$$= (x^3+6x^2-9x-14)(x^3+6x^2-13x+6)$$
$$= (x+1)(x-2)(x+7)(x-1)(x^2+7x-6),$$

therefore &c.

[It is worth remarking that, since the sums of the odd and even coefficients are each zero, the expression Σ is divisible by x^2-1.]

9652. (A. E. THOMAS.)—If

$$1+l\,.\frac{p-m}{m+1}\,.\,\frac{r+l}{n+l}+\frac{l(l-1)}{2!}\,.\,\frac{(p-m)(p-m-1)}{(m+1)(m+2)}\,.\,\frac{(r+l)(r+l-1)}{(n+l)(n+l-1)}+\&c.$$
$$\equiv f(n,\,r,\,p,\,m),$$

prove that $$\frac{f(n,\,r,\,p,\,m)}{f(p,\,m,\,n,\,r)}=\frac{p+l!}{m+l!}\,\frac{r+l!}{n+l!}\,\frac{m!}{p!}\,\frac{n!}{r!},$$

it being supposed that $n \nless r$, $p \nless m$.

Solution by the PROPOSER.

By equating coefficients of x^{n-r} in $(k+x)^n e^{k+x} = e^k\left[(k+x)^n e^x\right]$, and dividing throughout by $k^r_n C_r$, we have

$$1+k\frac{n+1}{r+1}+\frac{k^2}{2!}\,\frac{(n+1)(n+2)}{(r+1)(r+2)}+\text{etc.}$$
$$=e^k\left[1+k\frac{n-r}{r+1}+\frac{k^2}{2!}\,\frac{(n-r)(n-r-1)}{(r+1)(r+2)}+\text{etc.}\right]\ldots\ldots(a),$$

Writing down the corresponding series involving p and m, and eliminating e^k,

$$\left[1+k\frac{n+1}{r+1}+\frac{k^2}{2!}\,\frac{(n+1)(n+2)}{(r+1)(r+2)}+\text{etc.}\right]$$
$$\times\left[1+k\frac{p-m}{m+1}+\frac{k^2}{2!}\,\frac{(p-m)(p-m-1)}{(m+1)(m+2)}+\ldots\right]$$
$$=\left[1+k\frac{n-r}{r+1}+\frac{k^2}{2!}\,\frac{(n-r)(n-r-1)}{(r+1)(r+2)}+\ldots\right]$$
$$\times\left[1+k\frac{p+1}{m+1}+\frac{k^2}{2!}\,\frac{(p+1)(p+2)}{(m+1)(m+2)}+\ldots\right].$$

Equating coefficients of k^l on both sides of the above product, we find

$$\frac{n+l!}{r+l!}\,\frac{r!}{n!}f(n,\,r,\,p,\,m)=\frac{p+l!\;m!}{m+l!\;p!}f(p,\,m,\,n,\,r).$$

Several interesting series may be derived from (a). Thus, expanding e^k in the dexter and equating coefficients of k^l on both sides

$$\frac{n+l!}{r+l!}\,\frac{r!}{n!}=1+l\frac{n-r}{r+1}+\frac{l(l-1)}{2!}\,\frac{(n-r)(n-r-1)}{(r+1)(r+2)}+\text{etc.}=f(s,\,s,\,n,\,r).$$

Multiplying each side of (a) by $e^{(p/q)\,k}$, expanding and equating coefficients of k^l in the products,

$$1+l\frac{n+1}{r+1}\,\frac{q}{p}+\frac{l(l-1)}{2!}\,\frac{(n+1)(n+2)}{(r+1)(r+2)}\,\frac{q^2}{p^2}+\ldots$$
$$=\left[\frac{p+q}{p}\right]\left[1+l\frac{n-r}{r+1}\,\frac{q}{q+p}+\frac{l(l-1)}{2!}\,\frac{(n-r)(n-r-1)}{(r+1)(r+2)}\,\frac{q^2}{(q+p)^2}+\ldots\right].$$

Similarly two other series may be obtained from (a) by multiplying each side by (1) e^{-k}, (2) $e^{(-p/k-1)k}$, and proceeding as before.

APPENDIX I.

UNSOLVED QUESTIONS.

3501. (J. J. Walker, F.R.S.)—It is well known that the points of intersection of the three bisectors of sides of a plane triangle, of the three perpendiculars on sides, and of the three perpendiculars to sides, at their middle points, lie in the same straight line; prove that the three corresponding points in a spherical triangle cannot lie on the same great circle unless two sides are equal.

3504. (W. Siverly.)—In a circle $y^2 = 2rx - x^2$, find (1) the maximum ellipse inscribed between the ordinate, abscissa, and curve; and, (2) the locus of the centres of all such maximum ellipses that can be inscribed in a semicircle.

3510. (The Rev. T. P. Kirkman, M.A., F.R.S.)—The numbers of asymmetric heptagons on the 10-acral 10-edra, and on the 11-acral 10-edra, of which no one is the reflected image of another, are both between 50 and 60. Which is the greater number?

3517. (T. Mitcheson, B.A.)—If β, γ be the distances of the conjugate foci from the centre of a double convex lens whose thickness may be disregarded, for a ray of light diverging from Δ and converging to δ on the other side of the lens; ρ, ρ_1 the radii of the spherical surfaces; and α the distance of the focus to which the ray would converge, were the medium after the first refraction of uniform density; prove that

$$\alpha = \frac{\beta(\gamma+\rho_1)\rho + \gamma(\beta+\rho)\rho_1}{\beta(\gamma+\rho_1) - \gamma(\beta+\rho)}.$$

3532. (J. Griffiths, M.A.)—If r_1, r_2, r_3 be the radii, and δ_1, δ_2, δ_3 the distances between the centres of three given circles

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y = 0, \quad S_2 \equiv \&c. = 0, \quad S_3 \equiv \&c. = 0,$$

which meet in a common point; prove that the quartic

$$\Sigma(BC - F^2)S_1^2 + 2\Sigma(GH - AF)S_2S_3 = 0,$$

where $\qquad A = r_1^2 \sin^2\theta_1, \quad B = r_2^2 \sin^2\theta_2, \quad C = r_3^2 \sin^2\theta_3$

$$-2F = \delta_1^2 - r_2^2 - r_3^2 + 2r_2r_3 \cos\theta_2 \cos\theta_3,$$

with similar values for $-2G$, $-2H$, reduces to one of the form

$$(x^2+y^2)(x^2+y^2+2gx+2fy+c) = 0,$$

whatever be the values of the angles θ_1, θ_2, θ_3.

3557. (J. B. Sanders.)—Find the velocity and angle of elevation of a

ball, in order that it may be 100 feet above the ground at the distance of one quarter of a mile, and may strike the ground at the distance of one mile.

3559. (The Rev. T. P. Kirkman, M.A., F.R.S.)—Required the simplest functions, as to degree and number of terms, of *abcde*, and of *abcdef*, which have six values, and also the groups by whose substitutions they are invariable.

3569. (C. Taylor, M.A., D.D.)—Prove geometrically that the nine-point circle of a triangle touches the inscribed and escribed circles.

3589. (Omega.)—Sum the series

$$\frac{1}{2^4}+\frac{3^2}{2^6.3^2}+\frac{3^2.5^2}{2^8.3^2.4^2}+\&c.,\quad \frac{3^2}{2^3.3.4}+\frac{3^2.5^2}{2^5.3^2.4.5}+\frac{3^2.5^2.7^2}{2^7.3^2.4^2.5.6}+\&c.$$

3596. (The Rev. T. P. Kirkman, M.A., F.R.S.)—Required the groups by whose substitution the functions $ab+cd$, $ab+bd+dc+ca$, and $a^2b+d^2c+b^2a+c^2d$ are invariable.

3611. (W. Siverly.)—A cannon ball 1 foot in diameter, specific gravity 8 times that of water, was dropped from a bridge into the river below, and was 2 seconds falling to the water, and 2 seconds more in reaching the bottom of the river. Required the depth of the water.

3615. (J. B. Sanders.)—A body is projected from the top of a tower 200 feet high, at one angle of elevation of 60°, with a velocity of 50 feet. Find the range on the horizontal plane passing through the foot of the tower, and the time of flight.

3630. (Professor Hudson, M.A.)—The centre of a spherical mass is describing a parabola freely about the focus; the sphere expands as it approaches the focus, its radius varying inversely as the square of the distance, so that it traces out a parabolic tube of variable bore. According to what power of the distance does the rate of tracing out volume vary?

3640. (The Rev. A. F. Torry, M.A.)—The velocity of a particle freely describing a circle varies inversely as the ordinate to a fixed diameter: find the direction and law of the force.

3645. (W. Siverly.)—A uniform vertical pole standing on a horizontal plane begins to fall at the same instant that a monkey commences to climb it with a uniform velocity, and reaches the plane at the same instant the monkey reaches the top. Required the time occupied in falling, and the equation to the curve the monkey describes in space.

3646. (Hugh McColl, B.A.)—Find (by Sturm's Theorem or otherwise) the number of real roots of the equation

$$x^6-19x^5+90x^4+11x^3-99x^2+170x-70=0;$$

and determine the limits (contiguous integers or decimals) between which each real root is situated.

3647. (J. B. Sanders.)—A tube 30 inches long, and closed at one end and open at the other, was caused to descend in the sea with the open end downward, until the enclosed air occupied only one inch of the tube. How far did it descend?

3657. (Professor Minchin, M.A.)—Show that the force by which a particle describes the inverse (with respect to centre of force) of a given

curve is $\mu . \dfrac{r^6(r^2-2p\rho)}{p^3\rho}$, where r is the radius-vector to any point on the given curve, ρ and p the radius of curvature and perpendicular to a tangent at the point.

3658. (Professor Hudson, M.A.)—Trace the curve whose equation in trilinear coordinates is

$$y^2z^2\left(\frac{y}{q}+\frac{z}{r}\right)^2+z^2x^2\left(\frac{z}{r}+\frac{x}{p}\right)^2+x^2y^2\left(\frac{x}{p}+\frac{y}{q}\right)^2+\frac{x^2y^2z^2}{q^2}$$
$$=2xyz\left(\frac{x^3}{p^2}+\frac{y^3}{q^2}+\frac{z^3}{r^2}+\frac{zx(z+x)}{rp}+\frac{xy(x+y)}{pq}+\frac{yz(y+z)}{qr}\right).$$

3667. (J. B. Sanders.)—What number of degrees must a given volume of air be heated to double its elasticity?

3699. (Professor Hudson, M.A.)—A uniformly bright area, in the form of a quadrant of a circle, has one of its bounding radii in a given plane to which the plane of the quadrant is at right angles: prove that the curves of equal illumination on the given plane are found by eliminating ϕ between

$$r\sin\phi = a\cos(\theta-\phi) \quad\text{and}\quad a^2\cos^2\theta\,\epsilon^{2(\phi-\gamma)\tan\theta} = (r^2+a^2)\sin^2\phi.$$

3701. (W. Hogg, M.A.)—A particle of elasticity e falls from rest from a height a in a uniform medium, the resistance of which is kv^2, and impinging upon a perfectly hard horizontal plane, rises and falls alternately: find the whole space described before the motion ceases.

3707. (J. B. Sanders.)—A spherical air-bubble, having risen from a depth of 1000 feet in water, was one inch in diameter when it reached the surface. What was its diameter at the bottom?

3710. (Professor Crofton, F.R.S.)—Find the relation connecting any six simultaneous infinitesimal changes of length of the sides and diagonals of a rectangle. Apply the result to remove the ambiguity of the stresses produced in a rectangular frame of 4 joints united by 6 rods of the same material and section, by any system of 4 external balanced forces: in particular, when these four forces act in the directions of the diagonals; also when they act along a pair of opposite sides.

Any simple solution of the same problem for *any* quadrilateral frame, would be of practical value—whether by construction or calculation.

3729. (Professor Hudson, M.A.)—At noon, in latitude λ, the shadows upon a hill of the edges of a rectangular horizontal cloud are observed to be equally inclined at an angle γ to the horizon: regarding the hill as a plane inclined at an angle i to the horizon, with a north-westerly aspect, whose intersection with the horizon has an azimuth β east of north, find at what time of the year the observation took place. If $\lambda = 53^\circ 28'$, $\beta = 30^\circ = i$, and it was at midsummer, prove that

$$\gamma = \tan^{-1}(\tfrac{1}{4}\sqrt{3}).$$

3736. (T. Mitcheson, B.A.)—Find the area of the portion common to the two circles represented by the equation

$$x^4+y^4-16(x^3+xy^2)-32(y^3+x^2y)+2x^2(y^2+30)+32y(7x+6y)+144^2.$$

3737. (J. B. Sanders.)—To find the time in which a paraboloid of revolution whose altitude is h and parameter p, full of fluid, will empty itself through a small orifice at its vertex, its axis being vertical.

3762. (Artemas Martin, LL.D.)—If two dice be thrown horizontally at random into a circular box whose radius is equal to the diagonal of a face of a die; find the probability that only one of them rests on the bottom of the box.

3763. (D. Wickersham.)—Suppose A at the centre of a circular field of radius a, and B on the circumference. B walks round on the circumference at the rate of b miles an hour, and A walks directly towards B continually at the rate of c miles an hour, c being greater than b. Required the equation to the curve described by A, and the distance he must walk to overtake B.

3768. (J. Sanders.)—A conical vessel of base-radius r, and height h, is filled with fluid; find the time in which the fluid will descend through half the height, if an orifice be made at the vertex of the cone, when the axis is vertical.

3769. (The Editor.)—A sphere standing on a plane is cut in a great circle by a parallel plane, and is rolled on a straight line along the first plane. Find the surface, volume, and evolute of the part of the figure above the second plane formed by the great circle.

3777. (The Rev. T. P. Kirkman, M.A., F.R.S.)—Let

$$A = a_1 + a_2 + \ldots + a_n,$$

a_1, a_2, ... being any positive numbers, and $R \overline{\gtrless} A + n$. Let n groups of $a_1, a_2, \ldots a_n$ consecutive angles of an R-gon be marked, each group separated from the next by an interval of one or more unmarked angles. Let D be the number of ways in which d diagonals can be drawn in the R-gon, so that one at least shall be drawn from each of the A-marked angles, none crossing another. It is required to find the number D, which is independent of the order of the groups and of the magnitudes of the intervals.

3792. (G. S. Carr, M.A.)—A tube of smaller bore is joined to the open end of a common wheel-barometer tube, and alcohol is then poured in upon the mercury. Taking area of upper surface of mercury $= a$, that of the alcohol $= c$, area of common surface $= b$, sp. gr. of mercury $= m \times$ sp. gr. of alcohol; deduce the variation in the level of (c) corresponding to a variation (x) in the ordinary barometric column of mercury.

3796. (W. Siverly.)—A boy weighing a pounds climbs to the top of a slender conical hickory sapling b feet high, and c inches in diameter at the ground, and swings himself over, finding that his weight is just sufficient to bring the top of the sapling to the ground without breaking any part of it. Required the equation to the curve described by the sapling, and the distance from the foot where the top touches the ground.

3797. (Dr. S. H. Wright, M.A.)—A person sends a publisher money, and directs his paper to be sent as long as the money and its interest lasts. The interest is 7 per cent. *simple* interest, and price of the paper is 1 dol. 50 c. a-year in advance. Had *compound* interest been agreed upon, the paper could have been sent one year longer. Find what sum of money was sent, and how long the paper will be continued?

3798. (Artemas Martin, LL.D.)—Supposing the earth to be an oblate spheroid whose semi-axes are a and b; find the mean distance of its centre from all points in its surface.

3799. (J. B. Sanders.)—Find the equation of the curve described by the extremities of a horizontal diameter of the air-bubble of Quest. 3707, supposing its centre to move in a vertical line.

3804. (Professor Crofton, F.R.S.)—If five forces act along the sides of any convex pentagon, show that they cannot make equilibrium unless three act one way and two the opposite way; and that the two latter must not act along two adjacent sides of the pentagon.

3805. (S. Roberts, M.A.)—An angle ABC moves with the arm AB touching a curve U of the order m, and the arm CB touching a curve V of the order m', both curves being in the same plane. The locus of the point P rigidly connected with the angle is of the order $4mm'$; but if U and V coincide, the order is $4m(m-1)$. What reduction takes place if P coincides with B? And if U, V have double points and cusps, what is the effect on the locus?

3821. (W. Siverly.)—A very small bar of matter is movable about one extremity, which is fixed halfway between two centres of force, attracting inversely as the square of the distance. Find the positions of the equilibrium of the bar, and determine their nature.

3823. (Dr. S. H. Wright, M.A.)—When will the Mohammedan year commence on the 1st of January?

3836. (R. Tucker, M.A.)—The digits of a number of two figures are multiplied together, and if the resulting number consists of two figures these are also multiplied together, and the process continued until there is a single digit only; find the respective chances of this ultimate digit being (1) 5; or 2 (8); or (3) a square number; also (4) find the same for a number of three digits.

3837. (Dr. Hart.)—Find three numbers such that their sum is a square, the sum of their squares a square, and the sum of their cubes a square and an nth power.

3858. (R. Tucker, M.A.)—A blind man and his dog traverse the same curve path; the connecting string being kept stretched. Find (1) the curve which a stick carried by the dog at a constant inclination to the string always touches; and (2) when the two curves will be the same or similar.

3863. (The late M. Collins, B.A.)—Required an easy method for finding the remainder of $\frac{1}{73}$N, supposing N to be a very great whole number expressed by more than one or two hundred arithmetical figures.

3865. (ArtemasMartin,LL.D.)—The boundary lines of a quadrangular county ABCD are AB $= a$, BC $= b$, CD $= c$, DA $= d$, and the diagonal AC $= e$; find the most equitable position for the county seat, supposing the county uniformly populated.

3866. (W. Hogg, M.A.)—A particle is projected with a given velocity, towards a centre of force attracting inversely as the cube of the distance, in a medium of which the density varies inversely as the square of the distance from the centre of force; determine the velocity of the particle at any distance from the centre, the resistance for a given density varying as the square of the velocity.

3870. (W. Siverly.)—Find the centre of gravity of a bowl which is

a segment of a hollow sphere whose external and internal radii are R and r, the depth of the bowl being $(r-c)$.

3913. (R. Tucker, M.A.)—In Question 3858 discuss the curves obtained by supposing the dog and man to move in confocal conics.

3919. (Professor Hudson, M.A.) — A man's expenses exceed his income by £a per annum: he borrows at the end of every year enough to meet this, and, after the first year, to pay the interest on his previous borrowings, the rate of interest at which he borrows increasing each year in geometrical progression, whose common ratio is λ, till, at the end of n years, it is cent. per cent. What does he then borrow?

3935. (Professor Crofton, F.R.S.)—A plane disc is kept in equilibrium by tangential stresses acting round its contour (supposed convex). Show that in proceeding round the contour the stress must change in sign at least 4 times. If the *intensity* of the stress is constant all round the contour, show that if it change sign at 4 points only, those points must be the angles of a parallelogram. If it change sign at six points ABCDEF the parallelograms, three of whose vertices are ABC, DEF respectively, have their four vertices coincident.

3943. (Sir R. Ball, F.R.S.)—If a rigid body have four degrees of freedom, show that the body can be rotated about any line which intersects both of a certain pair of fixed lines.

3944. (J. J. Walker, F.R.S.)—If AD is the arc drawn from one angle A of a spherical triangle so as to bisect that angle and meet the opposite side BC in D, and if the arcs bisecting the other angles meet AD in O, prove that

$$\frac{\tan AD}{\tan AO} = \frac{2 \sin s \cos (s-a)}{\sin (b+c)}.$$

3953. (G. O'Hanlon.)—A string is unwound from a circle with a velocity v, while the circle rolls along a straight line with a velocity v'; determine $v : v'$ so that the evolute may be a closed curve.

3963. (The late T. Cotterill, M.A.)—Prove that—

1. In the conchoid of Nicomedes, a distance of a certain length resting on the curve and its asymptote passes either through the node of the curve or envelopes a four-cusped hypocycloid, the centre of which is the reflexion of the node to the asymptote.

2. A given distance resting on a hyperbola and either asymptote envelopes a parallel to a four-cusped hypocycloid, the bitangents of which are easily found.

3964. (The late M. Collins, B.A.)—Show that the Chaldean Saros, or period of 223 lunations (= 6585·3212 days), cannot be safely employed oftener than 49 or 50 times successively in predicting lunar eclipses.

3965. (Professor Crofton, F.R.S.)—The resultant normal pressure P between two blocks which meet at the plane joint AB is given, and passes through a given point X of the joint, AX being less than $\frac{1}{3}$AB. Supposing in all cases a *uniformly varying* distribution of the pressure, then, if (1) the joint does not admit of tension, the pressure at the joint only extends from A to a distance 3AX; but if (2) tension may exist, then the material is in tension at B, and in compression from A to some neutral

point. Show that the maximum intensity of compression is greater in case (1) than in case (2).

3973. (Professor Hudson, M.A.)—A flat circular disc of radius a is projected upon a rough horizontal table, which is such that the friction upon an element a is cV^3ma, where V is the velocity of the element, m the mass of a unit of area: find the path of the centre of the disc.

If the initial velocity of the centre of gravity and the angular velocity of the disc be u_0, ω_0, prove that the velocity u and angular velocity ω, at any subsequent time, satisfy the equation $\left(\dfrac{3u^2-a^2\omega^2}{3u_0^2-a^2\omega_0^2}\right)^2=\dfrac{u^2\omega}{u_0^2\omega_0}$.

3985. (Sir R. Ball, F.R.S.)—If a body have three degrees of freedom, show that the pitches of the screws about which the body can be twisted are proportional to the inverse squares of the parallel radii vectores intercepted between the centre and the surface of a certain hyperboloid.

4014. (C. H. Hinton.)—Let ABC be a triangle, and through a on BC draw a straight line cutting AC in b, AB in c. From centre b distance bC, draw a circle; from centre c distance cB, draw a circle cutting the other in P, P′. Bisect the external or internal angle of the triangle Pbc or P′bc by a straight line cutting ab in x. Let x' be a point on another straight line $ab'c'$ similarly obtained, and so on; then x, x', x''... lie on a straight line passing through A.

4015. (Dr. C. Taylor.)—Prove that any chord of a rectangular hyperbola subtends equal or supplementary angles at the ends of a perpendicular chord.

4017. (The late M. Collins, B.A.)—Prove that the indeterminate equation $(N^2+D)\,x^2-y^2=D^{2n+1}$ is always possible in whole numbers when N, D, n are whole numbers, and that x and y will be rational when N and D are so, n being any whole number.

4024. (J. Macleod, M.A.)—Prove (1) by geometrical construction that if a body be projected at an angle a to the horizon with the velocity due to gravity in 1″, its direction is inclined at an angle $\frac{1}{2}a$ to the horizon at the time $\tan\frac{1}{2}a$, and at an angle $\frac{1}{2}(\pi-a)$ at the time $\cot\frac{1}{2}x$. Also by the same method, show (2) that if the initial velocity of a projectile be given, the horizontal range is the same whether the angle of projection be $\frac{1}{4}\pi+a$ or $\frac{1}{4}\pi-a$; and show that the times of flight are inversely as the minimum velocities.

4029. (Artemas Martin, LL.D.)—If a penny be cut at random into two pieces, and the smaller piece placed on the larger; find the probability that the top piece will not fall off.

4030. (W. Barlow.)—Construct a quadrilateral when three sides are given and the fourth is trisected by perpendiculars from the opposite angles.

4040. (The late T. Cotterill, M.A.)—In hypo- or epicycloids, prove that the normal cuts the curve at the point where it is normal and touches its evolute in harmonic conjugates to its intersections with the concentric cuspidal circle; the ratio of harmonicism being that of the moveable circles.

4043. (H. McColl, B.A.)—Find the number and situation of the real roots of $x^4+4{\cdot}37162x^3-24{\cdot}9642358761x^2+34{\cdot}129226840859882x$

$-14{\cdot}63442007818570452204 = 0$, giving a near approximation to each, and employing whatever methods you consider most expeditious.

4047. (From Boole's *Differential Equations.*)—Show that the equation $(1-ax^2)\dfrac{d^2y}{dx^2}-bx\dfrac{dy}{dx}-cy=0$ is integrable in finite terms—

(1) if $\dfrac{b}{a}$ is an odd integer; (2) if $\left\{\left(1-\dfrac{b}{a}\right)^2+\dfrac{4c}{a}\right\}^{\frac{1}{2}}$ is an odd integer;

(3) if $\dfrac{b}{a}+\left\{\left(1-\dfrac{b}{a}\right)^2+\dfrac{4c}{a}\right\}^{\frac{1}{2}}$ or $\dfrac{b}{a}-\left\{\left(1-\dfrac{b}{a}\right)^2+\dfrac{4c}{a}\right\}^{\frac{1}{2}}$ is an even integer. [This is a generalized form of Question 3974, of which two solutions are given on pp. 33—35 of Vol. XIX.]

4049. (Artemas Martin, LL.D.)—Two uniform rods, of lengths a and b, rest with their ends against each other in a hemispherical bowl, of radius r and thickness c. The bowl rests on a horizontal plane, but is free to tip. Supposing the weight of the rods to be one $-n$th of the weight of the bowl; find the position of equilibrium of the bowl and rods.

4050. (W. Siverly.) — A bowl, which is the segment of a hollow sphere, whose external and internal radii are R and r, depth $r-c$, and weight W, contains a sphere of radius r' and weight P, which is attached to a weight Q by means of a string passing over the edge of the bowl. The bowl rests on a horizontal plane, but is free to tip. Find the positions of equilibrium of the sphere and bowl.

4052. (The Rev. W. Roberts, M.A.)—Let a polygon, whose sides are arcs of great circles be inscribed in a spherical equilateral hyperbola; also, let arcs of great circles be drawn connecting either extremity of each side with the point diametrically opposite to the other. Prove that if a side be denoted by s, and any of the last mentioned arcs by δ, $\Sigma\left(\tan\frac{1}{2}s\tan\frac{1}{2}\delta\right)=0$.

4053. (G. O'Hanlon.)—Find a superior limit to the number of games of whist that can be played.

4054. (T. Mitcheson, B.A.)—In a spherical triangle, if $t =$ the product of the tangents of the radii of the escribed circles, show that

$$\tan r = (n^4 \sin^{-2} st^{-1})^{\frac{1}{2}}.$$

4058. (A. Renshaw.)—Let ACB be a right-angled triangle, and CP the perpendicular on AB. Then, if circles be drawn on the three sides, and a tangent to that on AB be drawn through C, cutting the other two circles in E and F, prove (1) that CF shall be equal to CE; and (2) that the line FCE shall divide the exterior semicircles on AC, CB into four segments, the sum of the alternate pairs of which shall be respectively equal to the segments of the semicircle on AB made by AC, CB; also (3) deduce other properties of the figure.

4061. (R. Tucker, M.A.)—A heavy particle attached to an elastic string swings backwards and forwards in a vertical plane; find (1) the form of the curve the particle describes; (2) if the particle be disconnected at any instant, on what curve the focus of the subsequent parabolic path lies; and (3) what this last curve becomes when the string is inelastic.

APPENDIX II.

NOTES AND SOLUTIONS.

BY JOHN GRIFFITHS, M.A.

I. *Note on Invariantisers of Quantics.*

THE idea with respect to Invariantisers, as defined below, is, so far as I know, new, and was suggested to me by some results in my paper on the reduction of the differential expression $dt \div (t-\alpha \,.\, t-\beta \,.\, t-\gamma \,.\, t-\delta)^{\frac{1}{2}}$ to the standard form (see *Proceedings of the Lond. Math. Soc.*, Vol. XIV., p. 196, and Appendix to this Note).

As the method has recently been applied to simple Reciprocants by Mr. Leudesdorf (*Proc. Lond. Math. Soc.*, Vol. XVII., p. 210), and to Simultaneous Reciprocants by Mr. Berry (*Q. Journal*, Vol. XXIII., p. 291), a short account of it may be thought not unnecessary.

10300. (J. Griffiths. *Definitions.*)—(Binary Quantics).

1. An Invariant is taken to be a function of the elements $a_0, a_1, a_2 \dots a_n$ of a binary quantic,

$$I_n(a_0, a_1, a_2 \dots a_n, x, y) = a_0x^n + na_1x^{n-1}y + \frac{n\,.\,n-1}{1\,.\,2}a_2x^{n-2}y^2 + \dots + na_{n-1}xy^{n-1} + a_ny^n,$$

which is reduced to zero by one, or both, of the two operators,

$$\Omega = a_0\partial a_1 + 2a_1\partial a_2 + 3a_2\partial a_3 + \dots + na_{n-1}\partial a_n,$$

$$O = a_{n-1}\partial a_{n-1} + 2a_{n-1}\partial a_{n-2} + 3a_{n-2}\partial a_{n-3} + \dots + na_1\partial a_0\,;$$

i.e., an invariant here includes semi-invariants and full invariants.

2. In like manner, a Covariant is a function of $a_0, a_1, \dots a_n$, and the variables X, Y, which satisfies one or both of the relations,

$$Y\frac{dC}{dX} = \Omega C, \quad X\frac{dC}{dY} = OC\,;$$

i.e., C $(a_0, a_1 \dots a_n, X, Y)$ is either a semi-covariant or a full covariant. These are Professor CAYLEY's definitions.

In fact, throughout the note covariants and invariants generally mean semi-covariants and semi-invariants. Of course, a result with respect to one of the operators Ω can generally, *mutatis mutandis*, be applied to the other O. The above definitions may be extended so as to include quantics generally, *i.e.*, quantics of any degree in any number of variables. For instance, we may have an operator ω involving the coefficients of a quantic Q whose effect on Q, when the variables X, Y, Z ... are con-

sidered independent of ω, is the same as that of the operator $Y\partial X$ on Q, *i.e.*, we may have $$\omega Q = Y\frac{dQ}{dX}.$$

3. Invariantisers x and y of the above quantic are functions of the elements $a_0, a_1, a_2 \dots a_n$ which make the quantic an invariant as defined in 1.

4. Super-invariants are functions of the same elements which are reduced to zero by one of the operators Ω, O when applied a certain number of times. For example, since $\Omega\Omega(a_1a_2 - a_0a_3) = 0$, then $a_1a_2 - a_0a_3$ is a super-invariant of order 1, say.

(*Theorem* 1.)—If Ω, O be the two operators given above, then

$$\Omega I_n(a_0, a_1, a_1 \dots a_n, x, y) = \frac{dI_n}{dx}(\Omega x + y) + \frac{dI_n}{dy}\Omega y;$$

$$OI_n \dots = \frac{dI_n}{dx}Ox + \frac{dI_n}{dy}(Oy + x).$$

These results are proved without any difficulty.

As regards the operator Ω, we have

$$\Omega\left(a_0x^n + na_1x^{n-1}y + \frac{n.n-1}{1.2}a_2x^{n-2}y^2 + \dots + na_{n-1}xy^{n-1} + a_ny^n\right)$$

$$= \left(na_0x^{n-1} + n.n-1.a_1x^{n-2}y + \frac{n.n-1.n-2}{1.2}a_2x^{n-3}y^2 + \dots\right)\Omega x$$

$$+ x^n\Omega a_0 + nx^{n-1}y\Omega a_1 + \frac{n.n-1}{1.2}x^{n-2}y^2\Omega a_2 + \dots$$

$$+ (na_1x^{n-1} + n.n-1.a_2x^{n-2}y + \dots)\Omega y$$

$$= \frac{dI_n}{dx}(\Omega x + y) + \frac{dI_n}{dy}\Omega y, \text{ since } \Omega a_0 = 0,\ \Omega a_1 = a_0,\ \Omega a_2 = 2a_1 \dots.$$

Similarly with respect to O we have—

$$OI_n = \frac{dI_n}{dx}Ox + \frac{dI_n}{dy}(Oy + x).$$

10301. (J. Griffiths. *Theorem* 2.)—If $\Omega y = 0$ and $\Omega x + y = 0$, then I_n is reduced to zero by the operator Ω, or I_n is an invariant according to Definition 1.

This may be otherwise stated thus: if y is any invariant in the elements $a_0, a_1 \dots a_n$ which is annihilated by Ω, and x is a super-invariant of the same elements such that $\Omega x = -y$, then x and y are invariantisers of the quantic $I_n(a_0, a_1, \dots a_n . x, y)$. This follows at once from Theorem 1.

10303. (J. Griffiths. *Theorem* 3.)—If x and y are functions of $a_0, a_1 \dots a_n$ which satisfy the relations $\Omega y = 0$ and $\Omega x = -y$, then the quantics

$$\frac{dI_n}{dx}, \frac{d^2I_n}{dx^2}, \frac{d^3I_n}{dx^3} \dots,$$

are each invariantised by x and y.

This also follows immediately from Theorem 1. In fact, since

$$\frac{dI_n}{dx} = \frac{d}{dx}\left(a_0x^n + na_1x^{n-1}y + \frac{n.n-1}{1.2}a_2x^{n-2}y + \dots\right)$$

$$= n(a_0x^{n-1} + n-1.a_1x^{n-2}y + \dots) = nI_{n-1},$$

we have from theorem 1—

$$\Omega \frac{dI_n}{dx} = n\Omega I_{n-1} = n\left\{\frac{dI_{n-1}}{dx}(\Omega x + y) + \frac{dI_{n-1}}{dy}\Omega y\right\} = 0,$$

if $\Omega y = 0$ and $\Omega x + y = 0$. Similarly $\frac{d^2I_n}{dx^2}$, $\frac{d^3I_n}{dx^3}$, ... are invariantised by the said functions x, y.

It is clear that this will also be true with regard to the series of quantics I_{n+1}, I_{n+2} ... continued indefinitely.

10304. (J. Griffiths. *Theorem* 4.)—If $\Omega y = 0$ and $\Omega x + y = 0$, the quantics

$$\frac{dI_n}{dy}, \frac{d^2I_n}{dxdy}, \frac{d^2I_n}{dy^2}, \frac{d^3I_n}{dx^2dy}, \frac{d^3I_n}{dx\,dy^2}, \frac{d^3I_n}{dy^3}, \text{ \&c.}$$

are super-invariants.

1. If $\Omega y = 0$ and $\Omega x + y = 0$, then

$$\Omega \frac{dI_n}{dy} = \frac{dI_n}{dx}, \quad i.e., \quad \Omega\Omega \frac{dI_n}{dy} = \Omega \frac{dI_n}{dx} = 0,$$

from Theorem 3.

Taking $$nI_n = x\frac{dI_n}{dx} + y\frac{dI_n}{dy},$$

by Euler's theorem, we have

$$0 = -y\frac{dI_n}{dx} + y\Omega\frac{dI_n}{dy}, \text{ since } \Omega x = -y, \ \Omega y = 0, \ \Omega I_n = 0, \ \Omega\frac{dI_n}{dx} = 0,$$

therefore $$\Omega\frac{dI_n}{dy} = \frac{dI_n}{dx} = \text{an invariant.}$$

Hence if x, y be invariantisers of I_n, so that $\Omega x = -y$, and y = invariant, then $\frac{dI_n}{dy}$ and $-\frac{dI_n}{dx}$ are also invariantisers of the same quantic I_n.

2. Similarly, by Euler's theorem,

$$n \,.\, n-1 \,.\, I_n = x^2\partial_x^2 I_n + 2xy\,\partial_x\partial_y I_n + y^2\partial_y^2 I_n,$$

and hence operating with Ω we have, since $\Omega x + y = 0$, $\Omega y = 0$, $\Omega\partial_x^2 I_n = 0$ (see Theorem 3),

$$-2xy\,\partial_x^2 I_n + 2y\left\{x\Omega\,\partial_x\partial_y I_n - y\,\partial_x\partial_y I_n\right\} + y^2\Omega\,\partial_y^2 I_n = 0,$$

i.e., $$2x\left\{\Omega\,\partial_x\partial_y I_n - \partial_x^2 I_n\right\} + y\left\{\Omega\,\partial_y^2 I_n - 2\partial_x\partial_y I_n\right\} = 0.$$

Here y is an arbitrary function of the elements which has only to satisfy the partial differential equation $\Omega y = 0$, and we must therefore have $$\Omega\,\partial_x\partial_y I_n = \partial_x^2 I_n \text{ and } \Omega\,\partial_y^2 I_n = 2\partial_x\partial_y I_n.$$

In other words, $$\Omega\Omega\frac{d^2I_n}{dx\,dy} = \Omega\frac{d^2I_n}{dx^2} = 0,$$

and $$\Omega\Omega\Omega\frac{d^2I_n}{dy^2} = 2\Omega\Omega\frac{d^2I_n}{dx\,dy} = 0, \text{ or } \frac{d^2I_n}{dx\,dy}, \frac{d^3I_n}{dy^3},$$

are super-invariants of order 1 and 2 respectively. (See Definition 3.)

We may prove, in a similar manner, that if $\Omega y = 0$, $\Omega x + y = 0$, then

$$\Omega\frac{d^3I_n}{dx^2dy} = \frac{d^3I_n}{dx^3}; \quad \Omega\frac{d^3I_n}{dx\,dy^2} = 2\frac{d^3I_n}{dx^2dy}; \quad \Omega\frac{d^3I_n}{dy^3} = 3\frac{d^3I_n}{dx\,dy^2},$$

or that $\quad \Omega^2 \dfrac{d^3 I_n}{dx^2 dy} = 0; \quad \Omega^3 \dfrac{d^3 I_n}{dx\,dy^2} = 0; \quad \Omega^4 \dfrac{d^3 I_n}{dy^3} = 0;$

i.e., $\quad A_0 = \dfrac{d^3 I_n}{dx^3}, \quad A_1 = \dfrac{d^3 I_n}{dx^2 dy}, \quad A_2 = \dfrac{d^3 I_n}{dx\,dy^2}, \quad A_3 = \dfrac{d^3 I_n}{dy^3},$

where x, y are invariantisers satisfying the equations $\Omega y = 0$, $\Omega x + y = 0$, then $\quad \Omega A_0 = 0; \quad \Omega A_1 = A_0; \quad \Omega A_2 = 2A_1; \quad \Omega A_3 = 3A_2.$ The process may be continued with respect to the higher differential coefficients $\dfrac{d^4 I_n}{dx^3 dy}$, ... (see Appendix).

It thus appears that the functions A_3, ΩA_3, $\Omega^2 A_3$, $\Omega^3 A_3$, or A_3, $3A_2$, $6A_1$, $6A_0$, are connected with the coefficients of a semi-covariant; also $y = -A_0$ and $x = A_1$, for example, are invariantisers of I_n.

10305. (J. Griffiths. *Theorem* 5.)—If by means of invariantisers x, y of I_n which satisfy the equations $\Omega y = 0$, $\Omega x + y = 0$, we form a series of quantics,

$$\frac{d^2 I_n}{dx^2} X^2 + 2\frac{d^2 I_n}{dx\,dy} XY + \frac{d^2 I_n}{dy^2} Y^2,$$

$$\frac{d^3 I_n}{dx^3} X^3 + 3\frac{d^3 I_n}{dx^2 dy} X^2 Y + 3\frac{d^3 I_n}{dx\,dy^2} XY^2 + \frac{d^3 I_n}{dy^3} Y^3,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

then an invariant of any one of this series is an invariant of the original quantic I_n.

Taking, for instance, the cubic $A_0 X^3 + 3A_1 X^2 Y + 3A_2 XY^2 + A_3 Y^3$,

where $\quad A_0 = \dfrac{d^3 I_n}{dx^3}, \quad A_1 = \dfrac{d^3 I_n}{dx^2 dy}, \quad A_2 = \dfrac{d^3 I_n}{dx\,dy^2}, \quad A_3 = \dfrac{d^3 I_n}{dy^3},$

we have, from Theorem 4, an operator $\Theta = A_0 \partial A_1 + 2A_1 \partial A_2 + 3A_2 \partial A_3$,

where $\quad \Theta A_0 = 0, \quad \Theta A_1 = A_0 = \Omega A_1, \quad \Theta A_2 = 2A_1 = \Omega A_2,$

$\Theta A_3 = 3A_2 = \Omega A_3$; so that if $F(A_0, A_1, A_2, A_3) = f(a_0, a_1, \ldots a_n)$,

$$\Theta F = \frac{dF}{dA_0}\Theta A_0 + \frac{dF}{dA_1}\Theta A_1 + \frac{dF}{dA_2}\Theta A_2 + \frac{dF}{dA_3}\Theta A_3$$
$$= \frac{dF}{dA_0}\Omega A_0 + \frac{dF}{dA_1}\Omega A_1 + \ldots = \Omega F = \Omega f(a_0, a_1 \ldots),$$

and, consequently, $\Omega F = 0$ when $\Theta F = 0$, *i.e.*, if F is an invariant in A_0, A_1, A_2, A_3, it is also an invariant as regards the elements $a_0, a_1, a_2, \ldots a_n$.

10306. (J. Griffiths. *Theorem* 6.)—The results of theorems 4 and 5 may be otherwise stated thus: if the following functions of the elements $a_0, a_1 \ldots a_n$, viz.,

$$\frac{d^2 I_n}{dx^2}, \quad \frac{d^2 I_n}{dx\,dy}; \quad \frac{d^3 I_n}{dx^3}, \quad \frac{d^3 I_n}{dx^2 dy}, \ldots \text{ be formed, as above,}$$

from a quantic,

$$I_n(a_0, a_1, \ldots a_n, x, y) = a_0 x^n + n a_1 x^{n-1} y + \frac{n \,.\, n-1}{1 \,.\, 2} a_2 x^{n-2} y^2 + \ldots + a_n y^n,$$

by means of invariantisers which satisfy the equations $\Omega y = 0$ and

$\Omega x + y = 0$, then each of the series of quantics,

$$\frac{d^2 I_n}{dx^2} X^2 + 2\frac{d^2 I_n}{dx\,dy} XY + \frac{d^2 I_n}{dy^2} Y^2,$$

$$\frac{d^3 I_n}{dx^3} X^3 + 3\frac{d^3 I_n}{dx^2\,dy} X^2Y + 3\frac{d^3 I_n}{dx\,dy^2} XY^2 + \frac{d^3 I_n}{dy^3} Y^3,$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

is a semi-covariant of the quantic $(a_0, a_1, a_2 \dots a_n \between X, Y)^n$.

Take, for instance, the quantic $\frac{d^3 I_n}{dx^3} X^3 + \dots + \frac{d^3 I_n}{dy^3} Y^3$;

This may be written in the form,

$$\frac{\Omega^3 A_3}{1\,.\,2\,.\,3} X^3 + \frac{\Omega^2 A_3}{1\,.\,2} X^2Y + \Omega A_3 XY^2 + A_3 Y^3,$$

where $\qquad \Omega = a_0 \partial a_1 + 2a_1 \partial a_2 + \dots + na_n \partial a_{n-1},$

i.e., the quantic is a semi-covariant of $(a_0, a_1, a_2 \dots a_n \between X, Y)^n$.

For example, take $n = 3$, $x = a_1$, $y = -a_0$, so that

$$I_3 = a_0 x^3 + 3a_1 x^2 y + 3a_2 x y^2 + a_3 y^3,$$

$$\frac{d^2 I_3}{dx^2} = 6(a_0 x + a_1 y) = 0,$$

$$\frac{d^2 I_3}{dx\,dy} = 6(a_1 x + a_2 y) = 6(a_1^2 - a_0 a_2),$$

$$\frac{d^2 I_3}{dy^2} = 6(a_2 x + a_3 y) = 6(a_1 a_2 - a_0 a_3),$$

therefore the quantic $\quad 2(a_1^2 - a_0 a_2) XY + (a_1 a_2 - a_0 a_3) Y^2$

is, according to the theorem, a semi-covariant of a cubic which satisfies the relation

$$Y \frac{d}{dX} \{2(a_1^2 - a_0 a_2) XY + (a_1 a_2 - a_0 a_3) Y^2\}$$

$$= \Omega 2(a_1^2 - a_0 a_2) XY + \Omega (a_1 a_2 - a_0 a_3) Y^2;$$

i.e., since $\qquad \Omega(a_1^2 - a_0 a_2) = 0;$

$$\Omega(a_1 a_2 - a_0 a_3) = \left(a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + 3a_2 \frac{d}{da_3}\right)(a_1 a_2 - a_0 a_3) = 2(a_1^2 - a_0 a_2),$$

we have the identity $\quad (a_1 a_2 - a_0 a_3) Y^2 = (a_1 a_2 - a_0 a_3) Y^2,$

10307. (J. Griffiths. *Theorem* 7.)—If x and y be invariantisers of $I_n(x, y)$ which are rational integral functions of the elements $a_0, a_1, \dots a_n$, satisfying the equations $\Omega y = 0$, $\Omega x + y = 0$, then Ox, Oy are super-invariants of order 2 and 1 respectively, *i.e.*, $\Omega^3 Ox = 0$, $\Omega^2 Oy = 0$.

We have, according to a theorem due to Professor Sylvester,

$$\Omega O I_n - O\Omega I_n = \nu I_n, \text{ and } \Omega O y - O \Omega y = \mu y,$$

where μ and ν are mere numbers, so that, if I_n and y are invariants, as before, $\Omega O I_n = \nu I_n$, $\Omega O y = \mu y$; consequently, $\Omega^2 O I_n = 0$, and $\Omega^2 O y = 0$.

Again, $\Omega O x - O \Omega x = \lambda x$, where λ is a number; or $\Omega O x + O y = \lambda x$;

therefore $\qquad \Omega^2 O x + \Omega O y = \lambda \Omega x, \quad \Omega^2 O x + \mu y = -\lambda y;$

consequently $\qquad \Omega^3 O x = 0.$

10308. (J. Griffiths. *Theorem* 8.)—The results of the preceding theorems may be also utilised in the following way:—

If $\phi(\mathrm{A}, \mathrm{B}, \dots x, y)$, where A, B, ... are functions of the elements $a_0, a_1, \dots a_n$ of a quantic $\mathrm{I}_n = (a_0, a_1, \dots)(x, y)^n$, be a covariant or semi-covariant of I_n so that $y\dfrac{d\phi}{dx} = \Omega\phi$, then any pair of conjugate invariantisers x and y, such that $\Omega y = 0$ and $\Omega x + y = 0$, will also invariantise the covariant ϕ; *i.e.*, $\Omega\phi(\mathrm{A}, \mathrm{B}, \dots x, y) = 0$.

For example, let $n = 3$, and

$$\phi(\mathrm{A}, \mathrm{B}, \dots x, y) = (2a_1^3 - 3a_0a_1a_2 + a_0^2a_3)\,x^3 + 3\,(a_0a_1a_3 - 2a_0a_2^2 + a_1^2a_2)\,x^2y$$
$$+ 3\,(2a_1^2a_3 - a_1a_2^2 - a_0a_2a_3)\,xy^2 + (3a_1a_2a_3 - 2a_2^3 - a_0a_3^2)\,y^3$$

be a covariant of

$$\mathrm{I}_3\,(a_0, a_1, a_2, a_3)(x_1 y)^3 = a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3,$$

then if we take $y = -a_0$ and $x = a_1$, which satisfy the conditions $\Omega y = 0$ and $\Omega x + y = 0$, we have

$$\phi(\mathrm{A}, \mathrm{B}, \dots a_1, -a_0) = (2a_1^3 - 3a_0a_1a_2 + a_0^2a_3)\,a_1^3$$
$$-3\,(a_0a_1a_3 - 2a_0a_2^2 + a_1^2a_2)\,a_1^2a_0 + 3\,(2a_1^2a_3 - a_1a_2^2 - a_0a_2a_3)\,a_1a_0^2$$
$$-(3a_1a_2a_3 - 2a_2^3 - a_0a_3^2)\,a_0^3,$$

or $\quad \phi(\mathrm{A}, \mathrm{B}, \dots a_1, -a_0) = \tfrac{1}{2}(2a_1^3 - 3a_0a_1a_2 + a_0^2a_3)^2$
$$+ \tfrac{1}{2}a_0^2\,(a_0^2a_3^2 + 4a_0a_2^3 + 4a_1^3a_3 - 3a_1^2a_2^2 - 6a_0a_1a_2a_3);$$

i.e., $\qquad 2\phi =$ sum of two semi-invariants,

or $\qquad 2\phi + a_0^2 =$ sum of a semi-invariant and a full invariant of the original quantic I_3;

the latter being the discriminant $a_0^2a_3^2 + 4a_0a_2^3 + \dots$.

The theorem is at once proved.

$\phi(\mathrm{A}, \mathrm{B}, \dots x, y)$ is, *ex hypothesi*, a function of A, B, ... x, y, *i.e.*, of $a_0a_1a_2 \dots a_n\ x, y$, which satisfies the relation $y\dfrac{d\phi}{dx} = \Omega\phi$; but

$$\Omega\phi(\mathrm{A}, \mathrm{B}, \dots x, y) = \frac{d\phi}{dx}\,\Omega x + \frac{d\phi}{dy}\,\Omega y + \Omega\phi = \frac{d\phi}{dx}(\Omega x + y) + \frac{d\phi}{dy}\,\Omega y,$$

where $\Omega\phi$ means the function we get by applying Ω, and considering x and y to be constants. Hence

$$\Omega\phi(\mathrm{A}, \mathrm{B}, \dots x, y) = 0, \text{ if } \Omega x + y = 0 \text{ and } \Omega y = 0.$$

II. *Application of the Method to Ternary and other Quantics.*

The method is clearly applicable to ternary and other quantics. For example, let

$$\mathrm{I}_2\,(a, b, c, f, g, h, x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy.$$

Then $\quad \Omega\mathrm{I}_2\,(a, b, \dots x, y, z) = \dfrac{d\mathrm{I}_2}{dx}\,\Omega x + \dfrac{d\mathrm{I}_2}{dy}\,\Omega y + \dfrac{d\mathrm{I}_2}{dz}\,\Omega z + \Sigma x^2\Omega a + 2\Sigma yz\Omega f,$

or $$\Omega I_2(a, b, c, \ldots x, y, z) = \frac{dI_2}{dx}(\Omega x+y)+\frac{dI_2}{dy}\Omega y+\frac{dI_2}{dz}\Omega z$$

if the operator Ω is such that

$$\Sigma x^2\Omega a + 2\Sigma yz\Omega f = y\frac{dI_2}{dx} = 2y\,(ax+hy+gz),$$

i.e., if $\Omega a = 0, \quad \Omega b = 2h, \quad \Omega c = 0, \quad \Omega f = g, \quad \Omega g = 0,$

$$\Omega h = a \quad \text{or} \quad \Omega \equiv a\partial_h + g\partial_f + 2h\partial_b.$$

For instance, let $x = hf-bg, \quad y = gh-af, \quad z = 0,$

then $$\Omega x = af+gh-2hg = af-gh = -y,$$

$$\Omega y = ag-ga = 0, \quad \Omega z = 0;$$

Hence $I_2(a, b, c, f, g, h, hf-bg, gh-af, 0)$ is an invariant,

where $$I_2 = ax^2+2hxy+by^2$$
$$= a\,(hf-bg)^2+2h\,(hf-bg)\,(gh-af)+b\,(gh-af)^2$$
$$= (ab-h^2)\,(af^2+bg^2-2fgh),$$

or, since $(a\partial_h+g\partial_f+2h\partial_b)(ab-h^2) = -2ah+2ha = 0,$

af^2+bg^2-2fgh is an invariant of the quantic.

Geometrically, the vanishing of this invariant gives the condition that an asymptote of the conic

$ax^2+2hxy+by^2+2fy+2gx+c = 0$ shall pass through the origin.

Similarly, we have operator w, whose effect on

$I_2(a, b, c, f, g, h, x, y, z)$ is as follows; viz.,

$$wI_2(a, b, c \ldots x, y, z) = \frac{dI_2}{dx}(wx+z)+\frac{dI_2}{dy}wy+\frac{dI_2}{dz}wz$$

if $$wI_2 = z\frac{dI_2}{dx} \text{ or } \Sigma x^2wa+2\Sigma yzwf = 2z\,(ax+hy+gz),$$

hence $wa = 0, \quad wb = 0, \quad wc = 2g, \quad wf = h, \quad wg = a, \quad wh = 0,$

or $$w = a\partial_g + h\partial_f + 2g\partial_c.$$

In fact, we have altogether six operators, viz.,

(1) $a\partial_h+g\partial_f+2h\partial_b$. (2) $a\partial_g+h\partial_f+2g\partial_c$.

(3) $2h\partial_a+f\partial_g+b\partial_h$. (4) $2f\partial_c+b\partial_f+h\partial_g$.

(5) $2g\partial_a+c\partial_g+f\partial_h$. (6) $2f\partial_b+c\partial_f+g\partial_h$.

The effect of (3), for example, is

$$wI_2(a, b, c, \ldots x, y, z) = \frac{dI_2}{dx}wx+\frac{dI_2}{dy}(wy+x)+\frac{dI_2}{dz}wz.$$

Hence $$wI_2(a, b, c, \ldots x, y, z) = 0,$$

if $wx = 0, \quad wy+x = 0, \quad wz = 0, \quad$ where $w \equiv 2h\partial_a+f\partial_g+b\partial_h$.

The semi-invariant af^2+bg^2-2fgh has already been shown to satisfy the condition $$(a\partial_h+g\partial_f+2h\partial_b)\,(af^2+bg^2-2fgh) = 0,$$

the invariantisers x, y, z, functions of a, b, c ..., being $x = hf - bg$, $y = gh - af$ and $z = 0$. Now if we apply the operator (3), viz.,

$$2h\partial_a + f\partial_g + b\partial_h,$$

we get $$(2h\partial_a + f\partial_g + b\partial_h)(hf - bg) = 0,$$

$$(2h\partial_a + f\partial_g + b\partial_h)(gh - af) = 2h(-f) + fh + bg = -hf + bg = -x;$$

$$(2h\partial_a + f\partial_g + b\partial_h)\, z = 0 \text{ since } z = 0.$$

In other words, the function $af^2 + bg^2 - 2fgh$ is annihilated by both the operators (1) and (3). This invariant, in fact, remains unaltered, if we substitute for X, Y, $X + \lambda Y$, $Y + \mu X$ in the quantic

$$aX^2 + bY^2 + cZ^2 + 2fYZ + 2gZX + 2hXY.$$

Let us take another example, viz., $x = hf - bg$, $y = gh - af$, $z = ab - h^2$. These invariantisers satisfy the relations $wx + y = 0$, $wy = 0$, $wz = 0$, where $w \equiv a\partial_h + g\partial_f + 2h\partial_b$, and the corresponding invariant is

$$(a, b, c, f, g, h)(x, y, z)^2 = a(hf - bg)^2 + \ldots + 2f(gh - af)(ab - h^2)$$

$$= (ab - h^2)(af^2 + bg^2 - 2fgh) + (ab - h^2)[(ab - h^2)c + 2(gh - af)f + 2(hf - bg)g]$$

$$= (ab - h^2)(abc + 2fgh - af^2 - bg^2 - ch^2).$$

Hence we have an invariant $I = abc + 2fgh - af^2 - bg^2 - ch^2$, which, in fact, is annihilated by each of the six operators given above. For instance, take the operator

$$w \equiv 2f\partial_c + b\partial_f + h\partial_g, \text{ then } wI = 2f(ab - h^2) + b(2gh - 2af) + h(2fh - 2bg)$$

or $$wI = 0.$$

This result is, of course, the well-known fact that in the conic whose equation is $$ax^2 + 2hxy + by^2 + 2fy + 2gx + c = 0,$$
the function $abc + 2fgh - af^2 - bg^2 - ch^2$ is unaffected by any change whatever of the coordinate axes.

The method is thus applicable to quantics of any degree and in any number of variables. Moreover, it shows us what the operators must be.

III. *Examples in illustration of the above Theorems.*

(A.) The quantities y and x which I have called Invariantisers, are not necessarily rational functions of the elements a_0, a_1, $a_2 \ldots a_n$. For example, in the quartic $(a_0, a_1, a_2, a_3, a_4 \between x, y)^4$, if we take $y = 1$, and α, β, γ, δ to be the roots of $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$, then $x = \dfrac{\alpha\delta - \beta\gamma}{\alpha - \beta - \gamma + \delta}$ satisfies the equation $\Omega x = -1$, since

$$\Omega\, \frac{\alpha\delta - \beta\gamma}{\alpha - \beta - \gamma + \delta} = -(\partial_\alpha + \partial_\beta + \partial_\gamma + \partial_\delta)\frac{\alpha\delta - \beta\gamma}{\alpha - \beta - \gamma + \delta} = -\frac{\delta - \gamma - \beta + \alpha}{\alpha - \beta - \gamma + \delta} = -1;$$

But, as I shall presently prove in the Appendix, $x = \dfrac{\alpha\delta - \beta\gamma}{\alpha - \beta - \gamma + \delta}$ is

a root of the cubic equation

$$(a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4)\,(a_0^2x^2+2a_0a_1x+a_1^2)$$
$$= a_0\,(a_0x^3+3a_1x^2+3a_2x+a_3)^2,$$

and thus cannot, in general, be a rational function of a_0, a_1, a_2, a_3, a_4.

(B.) As an example of y = invariant, and $\Omega x = -y$; take $y = a_0$ and $x = -a_1$.

Hence we derive the following invariants

$$I_1 = a_0x + a_1y = 0,$$
$$I_2 = a_0x^2 + 2a_1xy + a_2y^2 = a_0\,(a_0a_2 - a_1^2),$$
$$I_3 = a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3 = a_0\,(2a_1^3 - 3a_0a_1a_2 + a_0^2a_3),$$
$$I_4 = a_0x^4 + 4a_1x^3y + \ldots + a_4y^4 = a_0\,(-3a_1^4 + 6a_0a_1^2a_2 - 4a_0a_1a_3 + a_0^3a_4)$$
$$= a_0^2\,(a_0a_4 - 4a_1a_3 + 3a_2^2) - 3\,(a_0a_2 - a_1^2)^2,$$

and so on for higher quantics derived in the above manner.

Other results are obviously suggested by the process.

(C.) It is clear that the number of invariantisers x, y satisfying the conditions y = invariant and $\Omega x + y = 0$ is infinite. For example, take $y = 1$, $a_0x + a_1 = f\,(a_0a_2 - a_1^2)$, where f is an arbitrary function.

Again, as an example of invariantisers of a different class, take y = invariant, and x to be a root of $dI_n/dx = 0$, then we have, by theorem 1, $\Omega I_n = 0$, or I_n is an invariant. I may remark that I_n so invariantised will not necessarily be a rational function of the elements $a_0, a_1 \ldots a_n$. In general it will only be a factor of the discriminant of the quantic $I_n\,(x, y)$. For example, take the cubic

$$I_3\,(x, y) = a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3,$$

and put therein $y = 1$, so that $dI_3/dx = 0$ becomes $a_0x^2 + 2a_1x + a_2 = 0$,

or, say, $\quad x = \dfrac{A - a_1}{a_0}$, where $A = (a_1^2 - a_0a_2)^{\frac{1}{2}}$.

Consequently, $$I_3\left(\frac{A-a_1}{a_0},\ 1\right) = \frac{(A-a_1)^3}{a_0^2} + 3a_1\frac{(A-a_1)^2}{a_0^2} + 3a_2\frac{A-a_1}{a_0} + a_3$$

$$= \frac{A^3 + 3\,(a_0a_2 - a_1^2)\,A + B}{a_0^2} = \frac{B - 2A^3}{a_0^2};\ \text{ if } B = 2a_1^3 - 3a_0a_1a_2 + a_0^2a_3.$$

It thus appears that the discriminant of $I_3\,(x, y)$ is composed of the two factors $B + 2A^3$, and $B - 2A^3$.

(D.) As examples of super-invariants (see Def. 4), I notice the following, viz., when $x = a_1$, and $y = -a_0$.

$$I_1 = a_1x + a_2y = a_1^2 - a_0a_2 \equiv \text{invariant}.$$
$$I_2 = a_1x^2 + 2a_2xy + a_3y^2 = a_1^3 - 2a_0a_1a_2 + a_0^2a_3;\ \ \Omega\Omega I_2 = 0,$$
$$I_3 = a_1x^3 + 3a_2x^2y + 3a_3xy^2 + a_4y^4 = a_1^4 - 3a_0a_1^2a_2 + 3a_0^2a_1a_3 - a_0^3a_4,$$
$\Omega\Omega I_3 = 0$, and so on.

(E.) The following will illustrate the Oy and Ox formulæ. (See Theorem 6.)

Put $\quad y = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3, \quad -x = a_1^2a_2 + a_0a_2^2 - 3a_0a_1a_3 + a_0^2a_4,$

$$\Omega = a_0\,\partial a_1 + 2a_1\,\partial a_2 + 3a_2\,\partial a_3 + \ldots,\quad O = na_1\partial a_0 + (n-1)\,a_2\partial a_1 + (n-2)\,a_3\partial a_2 + \ldots$$

then

$\Omega y = -x$; $\quad Oy = (6-n)\, a_0a_1a_3 + (3n-6)\, a_1^2a_2 - 3\,(n-1)\, a_0a_2^2 + (n-3)\, a_0^2a_4,$

$$\Omega O y = (3n-6)\, y.$$

$Ox = (3n-2)\, a_1a_2^2 - (2n+2)\, a_1^2a_3 + (9-n)\, a_0a_1a_4 - (n+1)\, a_0a_2a_3 + (n-4)\, a_0^2a_5,$

$$\Omega\Omega O x = -2\,(3n-7)\, y.$$

IV. *Appendix.*

It appears from my paper on the reduction of

$$dt \div (t-a\,.\,t-\beta\,.\,t-\gamma\,.\,t-\delta)^{\frac{1}{2}} \text{ to the form } M\,\frac{dz}{(1-z^2\,.\,1-k^2z^2)^{\frac{1}{2}}}$$

that a root a of a biquadratic

$$a_0t^4 + 4a_1t^3 + 6a_2t^2 + 4a_3t + a_4 = 0,$$

may be expressed in terms of functions of the roots a, β, γ, δ by $a = x + y\phi$, where

$$x = \frac{a\delta - \beta\gamma}{a-\beta-\gamma+\delta};\quad y = \frac{\operatorname{cn} u_0 + \operatorname{dn} u_0}{1+\operatorname{dn} u_0} = \frac{\beta - a}{\beta - \delta};\quad y\phi = \frac{a-\beta\,.\,a-\gamma}{a-\beta-\gamma+\delta}.$$

(See *Proc.* L. M. S., Vol. XIV., p. 196).

This is, in fact, a mere identity $a = \dfrac{a\delta - \beta\gamma + a-\beta\,.\,a-\gamma}{a-\beta-\gamma+\delta}$.

Hence $\qquad a_0\,(x+y\phi)^4 + 4a_1\,(x+y\phi)^3 + \ldots + a_4 = 0;$

or, say, $\qquad A\phi^4 + 4B\phi^3 + 6C\phi^2 + 4D\phi + E = 0,$

if $\qquad A = a_0y^4;\quad B = y^3\,(a_0x + a_1);\quad C = y^2\,(a_0x^2 + 2a_1x + a_2);$

$D = y\,(a_0x^3 + 3a_1x^2 + 3a_2x + a_3);\quad E = a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4.$

But from the results of the paper in question it is seen that we may divide the four values of ϕ into two pairs ϕ_1, ϕ_2; ϕ_3, ϕ_4, so that

$$\phi_1\phi_2 = \frac{\operatorname{cn} u_0 + \operatorname{dn} u_0}{1+\operatorname{dn} u_0} \times \frac{\operatorname{cn} u_0 - \operatorname{dn} u_0}{1-\operatorname{dn} u_0} = -\frac{k'^2}{k^2} = \phi_3\phi_4.$$

In other words, the biquadratic in ϕ is capable of being reduced to a reciprocal one.

The condition for this is easily seen to be $\dfrac{A}{E} = \dfrac{B^2}{D^2}$,

or $\qquad (a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4)\,(a_0^2x^2 + 2a_0a_1x + a_1^2)$

$$= a_0\,(a_0x^3 + 3a_1x^2 + 3a_2x + a_3)^2,$$

the cubic referred to in Ex. (A) supra.

Again, since $\phi = \dfrac{\beta - a}{\beta - \delta}$ and $y = \dfrac{\gamma - a\,.\,\beta - \delta}{a-\beta-\gamma+\delta}$ are functions of the roots which remain unaltered when $a+\lambda$, $\beta+\lambda$, $\gamma+\lambda$, $\delta+\lambda$ are written for a, β, γ, δ it follows that the coefficients A, B, C, D, E must be invariants. It was this that suggested to me the idea of invariantisers. The above process shows that the reduction of a biquadratic equation to the reciprocal form depends on the solution of a cubic.

Theorem 4.—As the results in this theorem are important, I append another proof of them.

By EULER's theorem we have $(n-1)\dfrac{dI_n}{dx} = x\dfrac{d^2 I_n}{dx^2} + y\dfrac{d^2 I_n}{dx\,dy}$,

and $(n-1)\dfrac{dI_n}{dy} = x\dfrac{d^2 I_n}{dx\,dy} + y\dfrac{d^2 I_n}{dy^2}$; $\therefore$ since $\Omega y = 0$, $\Omega x = -y$,

$$\Omega\frac{dI_n}{dx} = 0 \text{ (Theorem 3), we have } -y\frac{d^2 I_n}{dx^2} + y\Omega\frac{d^2 I_n}{dx\,dy} = 0,$$

or
$$\Omega\frac{d^2 I_n}{dx\,dy} = \frac{d^2 I_n}{dx^2}; \text{ also } x\Omega\frac{d^2 I_n}{dx\,dy} - y\frac{d^2 I_n}{dx\,dy} + y\,\Omega\frac{d^2 I_n}{dy^2}$$

$$= (n-1)\,\Omega\frac{dI_n}{dy} = (n-1)\frac{dI_n}{dx} = y\frac{d^2 I_n}{dx\,dy} + x\frac{d^2 I_n}{dx^2};$$

i.e.,
$$x\frac{d^2 I_n}{dx^2} - y\frac{d^2 I_n}{dx\,dy} + y\,\Omega\frac{d^2 I_n}{dy^2} = y\frac{d^2 I_n}{dx\,dy} + x\frac{d^2 I_n}{dx^2},$$

or
$$\Omega\frac{d^2 I_n}{dy^2} = 2\frac{d^2 I_n}{dx\,dy}.$$

Again, let $A_0 = \dfrac{d^3 I_n}{dx^3}$, $A_1 = \dfrac{d^3 I_n}{dx^2\,dy}$, $A_2 = \dfrac{d^3 I_n}{dx\,dy^2}$, $A_3 = \dfrac{d^3 I_n}{dy^3}$,

then
$$(n-2)\frac{d^2 I_n}{dx\,dy} = xA_1 + yA_2 \quad\ldots\ldots\ldots\ldots(1),$$

$$(n-2)\frac{d^2 I_n}{dy^2} = xA_2 + yA_3 \quad\ldots\ldots\ldots\ldots(2),$$

$$(n-2)\frac{d^2 I_n}{dx^2} = xA_0 + yA_1 \quad\ldots\ldots\ldots\ldots(3).$$

Now $\Omega\dfrac{d^2 I_n}{dx^2} = 0$ (Theorem 3), and $\Omega x = -y$, $\Omega y = 0$,

therefore, from (3), $\qquad -yA_0 + y\Omega A_1 = 0, \quad \Omega A_1 = A_0;$

also, from (1), $\qquad (n-2)\,\Omega\dfrac{d^2 I_n}{dx\,dy} = -yA_1 + x\Omega A_1 + y\Omega A_2;$

but, from above, $\qquad \Omega\dfrac{d^2 I_n}{dx\,dy} = \dfrac{d^2 I_n}{dx^2},$

$$\therefore (n-2)\frac{d^2 I_n}{dx^2} = -yA_1 + x\Omega A_1 + y\Omega A_2,$$

$$xA_0 + yA_1 = -yA_1 + xA_0 + y\Omega A_2, \quad i.e., \ \Omega A_2 = 2A_1.$$

Lastly, from (2), $\qquad (n-2)\,\Omega\dfrac{d^2 I_n}{dy^2} = -yA_2 + x\Omega A_2 + y\Omega A_3,$

$$2(n-2)\frac{d^2 I_n}{dx\,dy} = -yA_2 + 2xA_1 + y\Omega A_3,$$

$$2(xA_1 + yA_2) = -yA_2 + 2xA_1 + y\Omega A_3,$$

or
$$\Omega A_3 = 3A_2.$$

And so on for the higher differential coefficients.

V. *Solutions of Old Questions.*

3532. (J. Griffiths, M.A.)—If r_1, r_2, r_3 be the radii, and δ_1, δ_2, δ_3 the distances between the centres of three given circles

$$S_1 \equiv x^2+y^2+2g_1x+2f_1y=0,\quad S_2 \equiv \&c. = 0,\quad S_3 \equiv \&c. = 0,$$

which meet in a common point; prove that the quartic

$$\Sigma\,(BC-F^2)\,S_1^2+2\Sigma\,(GH-AF)\,S_2S_3=0,$$

where $\quad A = r_1^2\sin^2\theta_1,\quad B = r_2^2\sin^2\theta_2,\quad C = r_3^2\sin^2\theta_3$

$$-2F = \delta_1^2-r_2^2-r_3^2+2r_2r_3\cos\theta_2\cos\theta_3,$$

with similar values for $-2G$, $-2H$, reduces to one of the form

$$(x^2+y^2)(x^2+y^2+2gx+2fy+c)=0,$$

whatever be the values of the angles θ_1, θ_2, θ_3.

Solution.

The equation of the quartic, which represents a pair of circles each cutting three given circles S_1, S_2, S_3 at the respective angles θ_1, θ_2, θ_3, was given by me in a paper published in the *Proceedings of the London Mathematical Society* in 1871. When, as a particular case of the general theorem, the three circles S_1, S_2, S_3 meet in a point, we may take

$$-g_1 = r_1\cos a_1,\quad -f_1 = r_1\sin a_1;\quad -g_2 = r_2\cos a_2,\quad -f_2 = r_2\sin a_2;$$

$$-g_3 = r_3\cos a_3,\quad -f_3 = r_3\sin a_3;$$

and prove that x^2+y^2 is a factor of the expression

$$\Sigma\,(BC-F^2)\,S_1^2+2\Sigma\,(GH-AF)\,S_2S_3.$$

Writing $x=\rho$ and $y=i\rho$, where $i=(-1)^{\frac{1}{2}}$, it is easily seen that S_1, S_2, S_3 become proportional to $r_1(\cos a_1+i\sin a_1)$, $r_2(\cos a_2+i\sin a_2)$, $r_3(\cos a_3+i\sin a_3)$ respectively; also, if $\cos\theta_1=a$, $\cos\theta_2=b$, and $\cos\theta_3=c$, we find without much difficulty that

$$\Sigma\,(BC-F^2)\,S_1^2+2\Sigma\,(GH-AF)\,S_2S_3$$

is proportional to

$$\Sigma A_1(\cos 2a_1+i\sin 2a_1)+2\Sigma F_1\left\{\cos(a_2+a_3)+i\sin(a_2+a_3)\right\},$$

where $\quad A_1 = \sin^2(a_2-a_3)-b^2-c^2+2\cos(a_2-a_3)\,bc,$

$$F_1 = \cos(a_3-a_1)\cos(a_1-a_2)-\cos(a_2-a_3)+\cos(a_2-a_3)\,a^2+bc$$
$$-\cos(a_1-a_2)\,ca-\cos(a_3-a_1)\,ab,$$

with similar values for B_1, C_1, &c., since

$$A = r_1^2(1-a^2),\quad F = r_2r_3\left\{\cos(a_2-a_3)-\cos\theta_2\cos\theta_3\right\},\ \&c.$$

Lastly, it may be verified that each of the expressions

$$\Sigma A_1\cos 2a_1+2\Sigma F_1\cos(a_2+a_3)\ \text{and}\ \Sigma A_1\sin 2a_1+2\Sigma F_1\sin(a_2+a_3)$$

vanishes, as it is found at once that the coefficients of a^2, bc, &c., and the terms independent of these arbitrary quantities, all vanish. In other words, x^2+y^2 must be a factor of the bicircular quartic

$$\Sigma\,(BC-F^2)\,S_1^2+2\Sigma\,(GH-AF)\,S_2S_3=0.$$

10309. (John Griffiths, M.A.)—Prove the following extension of FEUERBACH's theorem with regard to the nine-point circle:—If the right line joining a point (x, y, z) to its inverse (yz, zx, xy) passes through the centre of the circumcircle, then the pedal circle of the former pair touches the nine-point circle.

Note.

The condition that the pedal circle of a point (x, y, z) shall touch the nine-point circle was first given, in the form of a sextic, in my *Notes on the Geometry of the Plane Triangle*, published in 1867. At a subsequent date in the same year, I discovered, by means of some theorems in the book in question, that the sextic was really two coincident cubics [a note announcing this discovery was inserted on p. 35 of our Vol. VIII., for 1867]. The theorems referred to are those that immediately follow.

10310. (John Griffiths, M.A.)—If p, p' denote the pair of inverse points (x, y, z), (yz, zx, xy), and q, q' the pair (x', y', z'), $(y'z', z'x', x'y')$, then the sides of the triangle of reference and the quadrilateral pq, pq', $p'q$, $p'q'$ are seven tangents to the same conic.

Note.

The pedal circles of p and q and the director circle of the above seven tangent conic are coaxal. See *Notes on the Geometry of the Triangle*, pp. 31, 32.

10311. (John Griffiths, M.A.)—The director of any conic which touches the sides of the triangle of reference, and passes through the centre of the circum-circle has contacts of a similar species with the latter and the nine-point circle.

Note.

The extension of FEUERBACH's Theorem given in Quest. 10309 follows at once from the theorems in Quests. 10310-1. For if we take the point q to be the centre of the circum-circle and suppose p, p' to be in a line with q, then the seven-tangent conic will pass through q, and consequently its director circle will touch the pedal of q; *i.e.*, the nine-point circle. Hence by the second of the above theorems the pedal of p will also touch the nine-point circle.

APPENDIX III.

SOLUTIONS TO UNSOLVED QUESTIONS.

By James McMahon, B.A.

2879. (J. J. Walker, F.R.S.)—1. Show that the six values of the Anharmonic Ratio of a Steiner's triad of points on an ellipse and their fourth are given by the equation

$$16x^2y^2\left\{\lambda(\lambda-1)+1\right\}^3-27a^2b^2\lambda^2(\lambda-1)^2=0,$$

where (x, y) are the coordinates of the fourth point, through which the osculating circles at the other three pass, referred to the semi-axes ab; and (2) Determine the point (x, y) so that the four points may form a harmonic set.

Solution.

1. Defining points on an ellipse by means of the eccentric angle, the condition that the points α, β, γ, δ should lie on the same circle is $\alpha+\beta+\gamma+\delta=0$, or $2m\pi$ (Salmon, p. 229). Hence, if a circle osculate the ellipse at α and meet the curve again at δ, we have $3\alpha+\delta=0$, and

$$\tan\tfrac{1}{2}\delta=-\tan\tfrac{3}{2}\alpha=(\tan^3\tfrac{1}{2}\alpha-3\tan\tfrac{1}{2}\alpha)/(1-3\tan^2\tfrac{1}{2}\alpha).$$

This is a cubic equation for $\tan\frac{1}{2}\alpha$ when δ is given. (There would be a similar equation for $\tan\alpha$, but it will be found more convenient to work with $\tan\frac{1}{2}\alpha$.) Hence there are three points on an ellipse whose osculating circles pass through a given point on the curve. We wish now to find the anharmonic ratio of this triad of points and their fourth point δ; and we shall first prove that the anharmonic ratio of any four points α, β, γ, δ on the ellipse is equal to that of four points taken on a line, and determined by the four distances $\tan\frac{1}{2}\alpha$, $\tan\frac{1}{2}\beta$, $\tan\frac{1}{2}\gamma$, $\tan\frac{1}{2}\delta$, measured from any fixed point on the line. For the equations of the tangent lines at α, β, γ, δ are $x/a\cos\alpha+y/b\sin\alpha=1$, &c., and they meet the tangent $x=a$ in the four points whose ordinates are proportional to $(1-\cos\alpha)/\sin\alpha$, &c., that is, to $\tan\frac{1}{2}\alpha$, &c. But, for the triad of points in question, the tangents of the half-angles are found from the cubic

$$(x^3-3x)/(1-3x^2)=\tan\tfrac{1}{2}\delta,\ =m,\ \text{suppose},$$

or

$$x^3+3mx^2-3x-m=0;$$

then multiplying by the factor $x-m$, we introduce the fourth point in question, and obtain the quartic

$$x^4+2mx^3-3(1+m^2)x^2+2mx+m^2=0.$$

Of this equation we have now to form the well-known "anharmonic-ratio sextic," which for the general quartic is [see Burnside and Panton, p. 147, taken from Mr. WALKER's paper in the *Quarterly Journal of Mathematics*, Vol. x., pp. 55, 56]

$$4\,(I^3-27J^2)(\lambda^2-\lambda+1)^3 = 27I^3\lambda^2\,(\lambda-1)^2,$$

wherein $\quad I = a_0a_4-4a_1a_3+3a_2^2,$

$$J = a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_1^2a_4-a_2^3.$$

In the present case $\quad I = \tfrac{3}{4}\,(1+m^2)^2,$

$$J = -m^2\,(1+m^2)+\tfrac{1}{8}\,(1+m^2)^3,$$

therefore $\quad 4\,(I^3-27J^2) = 27m^2\,(1-m^4)^2,$

hence the sextic in λ, the anharmonic-ratio, is

$$64m^2\,(1-m^2)^2\,(\lambda^2-\lambda+1)^3 = (1+m^2)^4\,\lambda^2\,(\lambda-1)^2;$$

and, replacing m by $\tan\frac{1}{2}\delta$, $2m/(1+m^2)$ by $\sin\delta$ or y/b, and $(1-m^2)/(1+m^2)$ by $\cos\delta$ or x/a, the equation takes the required form

$$16x^2y^2\,(\lambda^2-\lambda+1)^3 = 27a^2b^2\,\lambda^2\,(\lambda-1)^2.$$

2. When the four points form a harmonic set $\lambda=-1$ is a root of the equation, then $4x^2y^2=a^2b^2$, therefore $1 = 4\sin^2\delta\cos^2\delta = \sin^2 2\delta$, hence one of the values of δ is 45°, and the required point is at the extremity of any of the equi-conjugate semi-diameters. Taking $\delta = 45°$, 405°, or 765°, the equation $3\alpha+\delta=0$ gives $\alpha=-15°$, $-135°$, or $-255°$, for the corresponding triad of points. To verify that the four points are harmonic, we may show that the tangents of half these angles are harmonic;

for $$\frac{(-\tan 67\tfrac{1}{2}+\tan 7\tfrac{1}{2})(-\tan 127\tfrac{1}{2}-\tan 22\tfrac{1}{2})}{(\tan 22\tfrac{1}{2}+\tan 7\tfrac{1}{2})(-\tan 127\tfrac{1}{2}+\tan 67\tfrac{1}{2})}$$

$$= \frac{-\sin 60^\circ-\sin 150}{\sin 30^\circ-\sin 120} = -1.$$

3139. (J. J. WALKER, F.R.S.)—What two relations must hold among Dr. Salmon's invariants A, B, C of the sextic $(a\ldots g)(x,y)^6$ when it is a perfect square?

Solution.

Any binary sextic that is a complete square must have the form

$$(lx+my)^2\,(l'x+m'y)^2\,(l''x+m''y)^2;$$

then if we use the substitutions $lx+my=X$, $l'x+m'y=Y$, the factor $l''x+m''y$ will take the form $pX+qY$, and the sextic will transform into $X^2Y^2\,(pX+qY)^2$, that is, into

$$p^2X^4Y^2+2pqX^3Y^3+q^2X^2Y^4;$$

thus, denoting the new coefficients by a', $6b'$, $15c'$, $20d'$, $15e'$, $6f'$, g', we have a', b', f', g' all $=0$, and also $15c'=p^2$, $10d'=pq$, $15e'=q^2$, whence $9c'e'=4d'^2$. But, when a', b', f', $g'=0$ the invariants A, B, C become

$$15c'e'-10d'^2,\quad c'^2e'^2-3c'd'^2e'+d'^4,\quad -8c'^3e'^3-39c'^2d'^2e'^2+36c'd'^4e'-8d'^6,$$

(Salmon, p. 263); hence we may obtain one condition by eliminating h, k

between the following three equations (in which, for brevity, h and k are written for $c'e'$ and d'^2):

$$\tfrac{1}{5}\mathrm{A} = 3h-2k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\mathrm{B} = h^2-3hk+k^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$-\tfrac{1}{4}\mathrm{C} = 2h^3+\tfrac{39}{4}h^2k-9hk^2+2k^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3);$$

we get from (1) and (2) $\quad (\tfrac{1}{5}\mathrm{A})^2-4\mathrm{B} = 5h^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$

and from (1), (2), (3) $\quad \tfrac{1}{5}\mathrm{AB}-\tfrac{1}{4}\mathrm{C} = 5h^3-\tfrac{5}{4}h^2k,$

$$= \tfrac{1}{8}(25h^3+\mathrm{A}h^2), \text{ from (1)},$$

therefore $\quad (8\mathrm{AB}-10\mathrm{C}-5\mathrm{A}h^2)^2 = (25h^2)^3,$

then substituting for $5h^2$ from (4), and clearing of fractions, we get

$$(\mathrm{A}^3-300\mathrm{AB}+250\mathrm{C})^2 = 5(\mathrm{A}^2-100\mathrm{B})^3.$$

This is the invariant condition (besides the vanishing of the discriminant) that is fulfilled by the coefficients of the given sextic when it has *two* square factors. The remaining condition for a *third* square factor is obtained from the relation $9c'e' = 4d'^2$ mentioned above; this gives $9h = 4k$, which, combined with (1) and (4), easily leads to the invariant condition

$$11\mathrm{A}^2+900\mathrm{B} = 0.$$

The relation $9h = 4k$ might have been used to simplify the first condition, for equations (1), (2), (3) would have become

$$\mathrm{A} = -\tfrac{15}{2}h, \quad \mathrm{B} = -\tfrac{11}{16}h^2, \quad \mathrm{C} = -\tfrac{74}{16}h^3,$$

giving the two conditions in the form

$$\left(\frac{\mathrm{A}}{15}\right)^6 = \left(\frac{-4\mathrm{B}}{11}\right)^3 = \left(\frac{\mathrm{C}}{37}\right)^2.$$

3420. (J. J. Walker, F.R.S.)—By what linear substitutions may $(a, b, c \between xy)^2$ and $(a', b', c' \between xy)^2$ be transformed simultaneously into $(\mathrm{A, B, C} \between xy)^2$ and [to a factor] $(\mathrm{A}, -\mathrm{B}, \mathrm{C} \between xy)^2$ respectively?

Solution.

Let $\quad ax^2+2bxy+cy^2 \equiv \mathrm{AX}^2+2\mathrm{BXY}+\mathrm{CY}^2,$

$$a'x^2+2b'xy+c'y^2 \equiv \rho(\mathrm{AX}^2-2\mathrm{BXY}+\mathrm{CY}^2),$$

then, taking invariants, we have

$$\mathrm{AC}-\mathrm{B}^2 = \mu^2(ac-b^2), \quad \rho^2(\mathrm{AC}-\mathrm{B}^2) = \mu^2(a'c'-b'^2),$$

where μ is the modulus of transformation, therefore

$$\rho^2 = (a'c'-b'^2) : (ac-b^2).$$

This determines ρ. Again, we have

$$(\rho a-a')x^2+2(\rho b-b')xy+(\rho c-c') = 4\rho\mathrm{BXY},$$

hence X, Y (with constant multipliers) are the factors of the left-hand member, and are therefore known in terms of x, y. Thus the required substitution is determined. *E.g.*, let

$$x^2+2xy+2y^2 \equiv \mathrm{AX}^2+2\mathrm{BXY}+\mathrm{CY}^2,$$

$$x^2+4xy+8y^2 \equiv \rho(\mathrm{AX}^2-2\mathrm{BXY}+\mathrm{CY}^2),$$

therefore $\quad AC-B^2 = 1 \,.\, \mu^2,\ \rho^2(AC-B^2) = 4\mu^2$, and $\rho = \pm 2$,

therefore $\quad x^2-4y^2 = 8BXY$, taking $\rho = +2$;

hence we may take $\quad X = x+2y, \quad Y = x-2y$,

therefore $\quad x = \frac{1}{2}(X+Y), \quad y = \frac{1}{4}(X-Y)$;

when these substitutions are made for x, y in the given quadratics, they become $\quad \frac{1}{8}(5X^2+2XY+Y^2), \quad \frac{1}{4}(5X^2-2XY+Y^2)$,

which have the form required.

If we had taken the negative value for ρ, we should have found

$$X = 3x+\left[4+(-20)^{\frac{1}{2}}\right]y, \quad Y = 3x+\left[4-(-20)^{\frac{1}{2}}\right]y.$$

Thus the problem has two solutions.

To give a geometrical interpretation to the result, we may observe that the factors of the right-hand members are of the form

$$(lX+mY)(l'X+m'Y), \quad \rho\,(lX-mY)(l'X-m'Y);$$

but the lines $lX \pm mY = 0$ are harmonic with the lines $X = 0,\ Y = 0$, and so are the lines $l'X \pm m'Y = 0$; hence the above algebraic problem corresponds to the following well-known geometric one: given a pair of lines L_1, L_2, and another pair M_1, M_2 concurrent with them; find another pair X, Y that shall be harmonic with L_1, M_1, and also with L_2, M_2.

It will be seen that the second algebraic solution gives the pair of lines that are harmonic with L_1, M_2, and also with L_2, M_1; for in the data there is no distinction between the two lines of each pair.

[This geometrical interpretation was given by Mr. Walker in a paper "On Corresponding Points in certain Involutions," published in the *London Mathematical Society's Proceedings*, Vol. III., p. 75.]

3252. (The late T. Cotterill, M.A.)—Prove the following theorems, in the enunciation of which a curve (simple or compound) of the order a is denoted by C_a:—

1. If of the $(a+b)\times p$ points of intersection of two curves C_{a+b} and C_p, $a\times p$ are on a curve C_a, the remaining $b\times p$ points are on a curve C_b.

2. If two curves C_a and C_p pass through a points on a curve C_k, then a curve C_{a+b} through the remaining $(a\times p-a)$ intersections will cut the curve C_p in $(bp+a)$ points lying on a curve C_{b+k}, which will cut the curve C_p again in $(k\times p-a)$ points on a curve C_k, which passes, or can be made to pass, through the a points from which we started.

2874. (The late T. Cotterill, M.A.)—Seven points on a cubic locus have an opposite point on the curve: *i.e.*, a variable cubic through seven given points cuts a fixed cubic through the same points, in two other points collinear with a point on the fixed cubic. Construct for the opposite point when the fixed cubic breaks up into a conic through five points and a line through two.

2754. (S. ROBERTS, M.A.)—Show that, if (a, a_1), (b, b_1) are points of contact of tangents from two points on a cubic curve, and (a, b), (a_1, b_1) have the same connective, then the four points lie on a conic which passes through their tangentials.

2419. (The late T. COTTERILL, M.A.)—1. If AA′, BB′, CC′ are the opposite intersections of a complete quadrilateral, an infinite number of cubics can be drawn through these points and another point D, touching DA, DA′ at A and A′. Amongst these cubics, there are two triads of straight lines, and four cubics having respectively a point of inflexion at B, B′, C, C′.

2. The locus of the intersection of tangents at B, B′ is the conic DAA′BB′; and of tangents at C, C′ is the conic DAA′CC′.

Give the reciprocal results when the class cubic degenerates.

2555. (The late Professor DE MORGAN.)—The following is a theorem of which an elementary proof is desired. It was known before I gave it in a totally different form in a communication (April, 1867) to the Mathematical Society, on the "Conic Octogram;" and the present form is as distinct from the other two as they are from one another. If I., II., III., IV. be the consecutive chord lines of one tetragon inscribed in a conic, and 1, 2, 3, 4 of another; the eight points of intersection of I. with 2 and 4, II. with 1 and 3, III. with 2 and 4, IV. with 1 and 3, lie in one conic section. A proof is especially asked for when the first conic is a pair of straight lines. There is, of course, another set of eight points in another conic, when the pairs 13, 24 are interchanged in the enunciation.

Solution.

(3252). 1. This is an extension of the well-known principle (Salmon, Art. 157) on which Prof. Sylvester's theory of "residuation" for the cubic is based.

We may use the theorem (proved Salmon, Art. 33) that: Every curve C_n drawn through $np-\frac{1}{2}(p-1)(p-2)$ given points on a curve C_p, where p is less than n, meets this curve in $\frac{1}{2}(p-1)(p-2)$ other fixed points. Now, of the remaining bp points in question, take a sufficient number to determine a curve C_b, viz., $\frac{1}{2}b(b+3)$, then there will still remain $bp-\frac{1}{2}b(b+3)$; and if this number does not exceed $\frac{1}{2}(p-1)(p-2)$, it is evident, by the theorem quoted, that the composite curve C_{a+b} (made up of C_a and C_b) will pass through all the remaining intersections of the given curve C_{a+b} with the curve C_p; and those intersections that do not lie on C_a will lie on C_b. Hence the proposition is true so long as

$$bp-\tfrac{1}{2}b(b+3) \not> \tfrac{1}{2}(p-1)(p-2),$$

i.e., $$b^2-(2p-3)b+(p-1)(p-2) \text{ not negative,}$$

or $$\left[b-(p-1)\right]\left[b-(p-2)\right] \text{ not negative,}$$

but this cannot be negative since b cannot lie between $p-1$ and $p-2$, therefore the proposition is proved.

2 Extending the words *residual* and *coresidual* (defined for the cubic in Salmon, Art. 158), it will now be seen by the method of Art. 159 that the

following theorem is true:—If two systems of points on a given curve C_p be coresidual, any other system that is a residual of one will be a residual of the other. Hence if a system S_1 be residual with S_2, and this system S_2 be residual with another system S_3, S_3 with S_4, S_4 with S_5, and so on, it is evident that S_1 will be residual to any system that has an even subscript, and coresidual with any system that has an odd subscript. From this we derive an easy solution of the question; for we have:—

(1) C_p met by C_a in the residual systems of a points and $ap-a$ points.

(2) C_p met by C_{a+b} in the residual systems of $ap-a$ points and $bp+a$ points.

(3) C_p met by C_{b+k} in the residual systems of $bp+a$ points and $kp-a$ points, hence the first system of a points is residual with the fourth system of $kp-a$ points; that is, they together make up the intersections of C_p with some curve C_k.

(2874). In this question the word "opposite" is equivalent to "coresidual."

First, let us understand by a *totality* of the nth order, on a curve C_p, a group of np points lying on a curve C_n. Thus two residual systems form a totality.

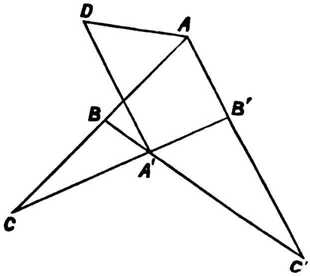
Now it will be seen, from the principles stated in the last answer, that *if in any totality we replace any group of points by a coresidual group we shall obtain another totality;* for instance, S_3+S_4 form a totality, and replacing S_3 by S_1 we have a totality S_1+S_4; but replacing S_3 by S_2, we have S_2+S_4, which is not a totality.

To apply this principle to the construction in question, let C_1, C_2, C_3, C_4, C_5 be the five points on the conic, and L_6, L_7 the two points on the line, then we may find two points l, c, residual to C_1, C_2, C_3, L_6, by letting C_1C_2 meet the line in l, and C_3L_6 meet the conic in c; the point c', in which lc meets the conic, is coresidual to C_1, C_2, C_3, L_6, and may take their place in finding residuals; thus we have now to find two points l', c'', residual to C_4, C_5, L_7, c', by letting C_4C_5 meet the line in l', and L_7c' meet the conic in c''; the point c''' in which $l'c''$ meets the conic is, then, the coresidual sought.

(2754). In this question the word "connective" is equivalent to "residual point." Let d, e be the points on the curve, from which the tangents are drawn; then the six points d, e, a_1, b_1, a_1, b_1 form a "totality" lying on the tangent lines da_1, eb_1; now, replacing a_1, b_1 by the co-residual system a, b, we have another totality of the second order d, e, a_1, b_1, a, b; hence these points lie on a conic, and this conic passes through the "tangentials" d, e.

[If A, B be the points from which the tangents are drawn, C the point where AB meets the cubic, then A, B, C are tangentials of a, b, c. The lines ab, a_1b_1 form a conic through the four given points cutting the cubic in two coincident points at c, hence every conic through aba_1b_1 meets the cubic again in two points collinear with C. Consider the conic aba_1b_1A; this must meet the cubic again in the same point as AC; that is to say, in B.]

(2419). 1. An infinite number of cubics can pass through the points A, A′, B, B′, C, C′, D, and touch DA at A, for these make only eight conditions. We shall now prove that each of these cubics touches DA′ at A′; for, on any one of them, the six points A, B, C, A, B′, C′ form a totality, and so do the six points B, A′, C′, B′, A′, C, hence the points A, A are co-residual with the points A′, A′, but D is residual to A, A, therefore D is residual to A′, A′, *i.e.*, DA′ touches the cubic at A′.

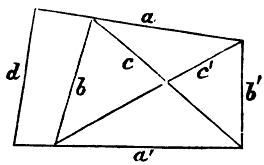


To this system of cubics evidently belong the line-triad DA, BA′C′, B′A′C, and also the triad DA′, ABC, AB′C′.

Again, there is one cubic of the system that fulfils the further condition of touching BB′ at B; and we shall prove that it has a point of inflexion at B′: for, on this cubic, B′ is residual to B, B, but, as above, B, B are co-residual to B′, B′, therefore B′ is residual to B′, B′, hence the tangent at B′ meets the cubic again in B′, *i.e.*, B′ is a point of inflexion on this cubic. Similarly there are cubics of the system that have points of inflexion at B, C, C′, respectively.

2. Take any one cubic of the system; it is met by any other in the totality D, A, A, A′, A′, B, C, B′, C′, therefore D, A, A′, B, B′ are residual with A, C, A′, C′, and hence co-residual with B, B, therefore the conic through D, A, A′, B, B′, and the tangent to the cubic at B, meet the cubic again in the same point; similarly, the tangent at B′ passes through this point; hence this fixed conic is the locus of the intersection of tangents at B, B′. Similarly for the other locus.

Reciprocating (1), we learn that, If a, a', b, b', c, c' be the six connectors of four points, and d any seventh line, an infinite number of class-cubics can touch these lines and have d for the line joining the points of contact that lie on a, a'; amongst these there are four having b, b', c, c', respectively for inflexional tangents.



The "degenerate" class cubics of the system are the reciprocals of the two line-triads above, viz., the two point-triads ad, $a'b$, $a'b'$, and $a'd$, ab, ab'; it will be seen that the first point-triad will satisfy all the requirements, if, for the condition that the point $a'd$ lies on the cubic envelope, be substituted the more general condition that $a'd$ is a point from which two coincident tangents can be drawn to the envelope; similarly for the second point-triad.

Reciprocating (2), we learn that the conic touching a, a', b, b', d is the envelope of the line joining the points where any curve of the system touches the lines b, b'; and, when the class-cubic degenerates into the first point-triad, the line whose envelope is sought becomes the line a', which, of course, touches the conic mentioned; similarly for the other point-triad. The envelope of the line joining the points of contact on the lines c, c' is obtained in the same way.

(2555). Denote the eight lines in order by $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta'$; then we have to prove that the points $\alpha\beta'$, $\alpha\delta'$, $\beta\gamma'$, $\beta\alpha'$, $\gamma\delta'$, $\gamma\beta'$, $\delta\alpha'$, $\delta\gamma'$ lie on a conic. Consider the two composite quartics $\alpha\gamma\alpha'\gamma'$ and $\beta\delta\beta'\delta'$: they intersect in 16 points of which eight lie on the first conic, therefore the other eight lie on another conic. This proves the proposition, even when the first conic is a composite of two lines.

2402. (R. Tucker, M.A.)—Prove that the locus of a point whose distance from its polar with reference to a given conic is equal to its distance from a given point is a quartic curve, which, when the conic becomes a circle, degenerates into a cubic curve.

Solution.

Taking the given point as origin, let the equation of the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

then the distance of a point $x'y'$ from its polar is

$$\frac{x'(ax' + hy' + g) + y'(hx' + by' + f) + gx' + fy' + c}{\left[(ax' + hy' + g)^2 + (hx' + by' + f)^2\right]},$$

and the equation of the required locus is

$$(ax^2 + 2hxy + by^2 + 2gx + 2fy + c)^2 = \left[(ax + hy + g)^2 + (hx + by + f)^2\right](x^2 + y^2),$$

or $\quad h^2(x^4 + y^4) + 2h(a-b)(xy^3 - x^3y) + \left[(a-b)^2 - 2h^2\right]x^2y^2 + \ldots = 0,$

which reduces to the third degree when $a = b$, and $h = 0$, therefore, &c.

It is evident also geometrically that the line at infinity is part of the locus, when the given conic is a circle.

2933. (A. Martin, LL.D.)—A boy walked across a horizontal turn-table while it was in motion at a uniform rate of speed, keeping all the time in the same vertical plane. The boy's velocity is supposed to be uniform with respect to his track on the table, and equal to m times the velocity of a point in the circumference of the table. Required the nature of the *curve* he described on the table, and the distance he walked while crossing it—

(1) When the motion of the table is *towards* him, (a) when $m > 1$, (b) when $m = 1$, and (c) when $m < 1$.

(2) When the motion of the table is *from* him, (a) when $m > 1$, (b) when $m = 1$, and (c) when $m < 1$.

Solution.

With the centre of the table as origin, let the axis of y be parallel to the vertical plane in question, and let abscissas be reckoned positive *towards* that plane; whose distance we may call d, and the radius of the

table a. The boy's motion relative to the table is not changed by supposing each to receive another motion, such as will bring the table to rest. We have then to compound the boy's angular velocity ω around the origin with his velocity parallel to the y-axis, to produce the resultant velocity $ma\omega$ in the required track; and since each velocity is proportional to the sine of the angle between the directions of the other two, we have

$$ma\omega : r\omega = \sin\theta : dx/ds,$$

therefore $$ds/dx = ma/r\sin\theta = ma/y,$$

therefore $$dy/dx = \mp (m^2a^2 - y^2)^{\frac{1}{2}}/y,$$

wherein the upper sign evidently corresponds to case (1) in which the track is convex towards its positive side, and the lower to case (2) in which it is convex towards the negative side; therefore

$$x - k = \pm (m^2a^2 - y^2)^{\frac{1}{2}},$$

k being a constant depending on d; hence the curve described is the circle $(x-k)^2 + y^2 = m^2a^2$, whose radius is ma, and whose centre is at the point $(k, 0)$. To determine k, we have, where the track cuts the circumference of the table,

$$d - k = \pm \left\{m^2a^2 - (a^2 - d^2)\right\}^{\frac{1}{2}} = \pm \left\{d^2 + (m^2-1)\,a^2\right\}^{\frac{1}{2}},$$

therefore $$k = d \mp \left\{d^2 + (m^2-1)\,a^2\right\}^{\frac{1}{2}}.$$

We next find v, the boy's velocity in the fixed vertical plane, from the proportion, $$v : r\omega = dr/ds : \mp dx/ds,$$

whence $$v = \mp \omega r\, dr/dx = \mp \tfrac{1}{2}\omega\, d(r^2)/dx,$$

but, from the equation of the circular track, we have

$$r^2 - 2kx = \text{constant}, \quad \text{and} \quad d(r^2)/dx = 2k,$$

therefore $$v = \mp k\omega.$$

It is now easy to distinguish between the different cases mentioned in the question.

In 1 (a), k is negative, the centre of the track is at the negative side of the origin, and v is positive; in 1 (b), k is zero, and the track coincides with the circumference of the table, also v is zero, and the boy remains at rest in space letting the table move under his feet; in 1 (c), k is positive, the centre of the track is at the positive side of the origin, and it will be seen that the track lies outside the circumference of the table, and that this requires a negative value of v, contrary to the supposition. In 2 (a), k is positive, the centre of the track is at the positive side of the origin, and v is positive; in 2 (b), k is positive and $= 2d$, and the track is the "reflexion," with respect to the vertical plane, of that part of the circumference which is at its positive side; in 2 (c), k and v are positive so long as $m \not< 1 - d^2/a^2$, when $m = 1 - d^2/a^2$, $k = d$, and the track is a semi-circumference, but when $m < 1 - d^2/a^2$, k and v are imaginary, showing that it is not possible for the boy to move in the vertical plane in such a way as to make m less than this.

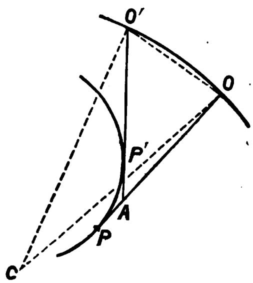
The length of the track is easily found to be $2ma \sin^{-1} \dfrac{(a^2 - d^2)^{\frac{1}{2}}}{ma}$, in all cases.

2971. (A. Martin, LL.D.)—Show that the solution of the famous "Curve of Pursuit Problem," when the object pursued moves uniformly in the circumference of a circle and the pursuer starts from the centre, can be made to depend upon the solutions of the differential equations,

$$d\theta = -\frac{dt}{n\cos\phi}, \quad tdt = r\left\{d(t\sin\phi) - ntd\phi\right\} \ldots\ldots\ldots\ldots (1, 2);$$

where r is the radius of the circle, t the distance the two objects are apart at any time during the motion, ϕ the angle t makes with a tangent to the circle, and θ the arc described by the pursued object from the commencement of the motion, supposing the pursuer to move n times as fast as the pursued.

Solution.



Let O, P be corresponding positions of the object and of the pursuer; then the geometrical condition to be satisfied is that the tangent at P to the pursuer's path shall always pass through O. Let O move to O′, and P to P′, in the same infinitesimal instant; then from the figure, which explains itself, we see that

$$dt = \text{O}'\text{P}' - \text{OP} = \text{O}'\text{A} - \text{OA} - (\text{AP} + \text{AP}')$$
$$= \text{OO}'\cos\phi - n\,.\,\text{OO}',$$

(neglecting infinitesimals of second order),

$$= d\theta\,(\cos\phi - n) \ldots\ldots\ldots\ldots (1).$$

Also, it is evident that

$$d\phi = \text{OCO}' - \text{OAO}' = \frac{d\theta}{r} - \frac{d\theta\sin\phi}{t},$$

therefore $\qquad rt\,d\phi = (t - r\sin\phi)\,d\theta,$

therefore, from (1), $\quad rt(\cos\phi - n)\,d\phi = dt\,(t - r\sin\phi),$

$$\therefore \quad tdt = r\,(t\cos\phi\,d\phi + \sin\phi\,dt - nt\,d\phi) = r\left\{d(t\sin\phi) - nt\,d\phi\right\} \ldots\ldots (2).$$

By elimination (if possible) we obtain a relation between t and θ, and one between ϕ and θ; then, assuming that at any instant the distance passed over by O, and the direction of its motion, are known, the relation between t and θ will give the distance between the two objects at the instant, while the relation between ϕ and $n\theta$ will evidently lead to the "intrinsic equation" of P's path.

8525. (G. S. Carr, M.A.)—If P, S are real points, coordinates (x, y) and (x', y'); and Q, R imaginary points, coordinates $(\alpha + i\alpha', \beta + i\beta')$ and $(\alpha - i\alpha', \beta - i\beta')$; show briefly that the *real* line which joins the imaginary points of intersection of the imaginary pairs of lines (PQ, SR), (PR, SQ) is identical with the line obtained by substituting unity for i in the imaginary coordinates, and drawing the five lines accordingly.

Note.

In reference to Mr. CARR's statement on my solution, given on pp. 165–6, of Vol. L., I would make the following remarks:—

A careful reading of Article 4761, on page 649 of Mr. CARR's *Synopsis*, will show that the argument in Lemmas II., III. is based on the assumption that the imaginary quantities $L \pm iM$, $L' \pm iM'$ become $L \pm M$, $L' \pm M'$ when i is changed into unity in the original coordinates. This implies that L, L′ are independent of the original i. But it will be seen that L, L′ (although real) have terms that depend on i^2, a quantity which changes sign when i is changed into unity.

In the notation of the geometric discussion, given on p. 165 of Vol. L., the points L_1, M_1 are not the graph-pair of the points L, M, and the line L_1M_1 does not coincide with LM.